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MECHANICS OF A NEAR NET-SHAPE STRESS-COATED MEMBRANE

Volume II of II
BOUNDARY VALUE PROBLEMS AND SOLUTIONS

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Final Report

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14. ABSTRACT

This report is the sequel to Volume I of the same title, in which asymptotic methods were used to derive theories that would aid in understanding the mechanical behavior of a stress-coated membrane. In Volume II we have applied those theories to a number of boundary value problems, obtaining generalizations of well-known solutions for a membrane, plate, or shell of a single material to solutions for the same structure, but now consisting of a multilayer-coated polymer material. Perhaps the most significant accomplishment of this work was the discovery of simple prescriptions for the coating stress that would maintain the shape of an initially parabolic coated membrane after removal from the mold upon which it was cast. These solutions involve linear combinations of Kelvin functions. Coating prescriptions are given for membrane laminates both with, and without, pressure and gravitational loads. The prescriptions are presently being used in the preliminary design of a near net-shaped stress-coated membrane, to be demonstrated hopefully in the near future.

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1 Introduction

A companion [1] to this Report discusses motivation for the analysis of near net-shape stress-coated membranes, and presents details of the derivations of four different theories of the mechanics of such a laminate using the method of asymptotic expansions. Our goals here are to present solutions of selected boundary value problems, and to compare the results with those obtained by the finite element (FE) method where available. Each of the next four Sections will be organized in a similar way, i.e., the governing equations for a given theory will first be restated from Reference [1], boundary conditions will then be imposed, and details of the solution procedure for each such set of boundary conditions will be presented. Finally, graphical comparisons will be made between the solutions of a given boundary value problem, and those using FE analysis.

With regard to nomenclature and notation, all problems presented here involve an N-layer structure consisting of a membrane substrate of thickness $h_N = h_s$ having $N-1$ coatings with thicknesses h_i, $i = 1, \ldots, N-1$, and *total* thickness $h = h_s + h_c$, where $h_c \equiv \sum_{k=1}^{N-1} h_k$ is the total *coating* thickness. This structure has a circular boundary of radius a. The reference configuration $\mathcal{S}$ of the coated membrane is defined by a mapping from a purely geometrical region $\mathcal{C}$, which we refer to as the reference placement. This mapping is illustrated for a single coating in Figure 2 of Reference [1], and reproduced below as Figure 1. As shown in the Figure, the coating and membrane thicknesses are assumed to be constant along any line parallel to the axis. The reference placement for $N-1$ coatings is illustrated in Figure 5 of Reference [1], and reproduced below as Figure 2. Note that the coatings are "on the bottom". The basis of this rather counterintuitive orientation is the *optical* convention that the direction of propagation of light, which strikes the coating first, coincides with the positive Z-direction (upward in our Figures). The coordinate $\xi = Z + h/2$ shown in Figure 2 is convenient for labeling the top and bottom surfaces of each layer, and we note from Figure 2 the important relation $\xi_i = \sum_{k=1}^{i} h_k$.

The following constants appear in the through-the-thickness integrals of the constitutive relations defining the stress resultants and stress couples, treated in detail in Appendix A of [1]:

$$\mathcal{N} = \sum_{i=1}^{N} h_i S_i, \qquad \mathcal{M} = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} h_i h_k (S_i - S_k), \tag{1.1}$$

$$A = \sum_{i=1}^{N} h_i Q_i, \qquad A_\nu = \sum_{i=1}^{N} h_i Q_i \nu_i, \tag{1.2}$$

$$B = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} h_i h_k (Q_i - Q_k), \qquad B_\nu = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} h_i h_k (Q_i \nu_i - Q_k \nu_k), \tag{1.3}$$

$$A_\Theta = \sum_{i=1}^{N} h_i G_i, \qquad B_\Theta = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} h_i h_k (G_i - G_k), \tag{1.4}$$

$$D = \frac{1}{12} \sum_{i=1}^{N} Q_i h_i \left[h_i^2 + 3(h - h_i)^2 - 12\xi_{i-1}(h - \xi_i) \right], \tag{1.5}$$

$$D_\nu = \frac{1}{12} \sum_{i=1}^{N} Q_i \nu_i h_i \left[h_i^2 + 3(h - h_i)^2 - 12\xi_{i-1}(h - \xi_i) \right], \tag{1.6}$$

$$D_\Theta = \frac{1}{12} \sum_{i=1}^{N} G_i h_i \left[h_i^2 + 3(h - h_i)^2 - 12\xi_{i-1}(h - \xi_i) \right], \tag{1.7}$$

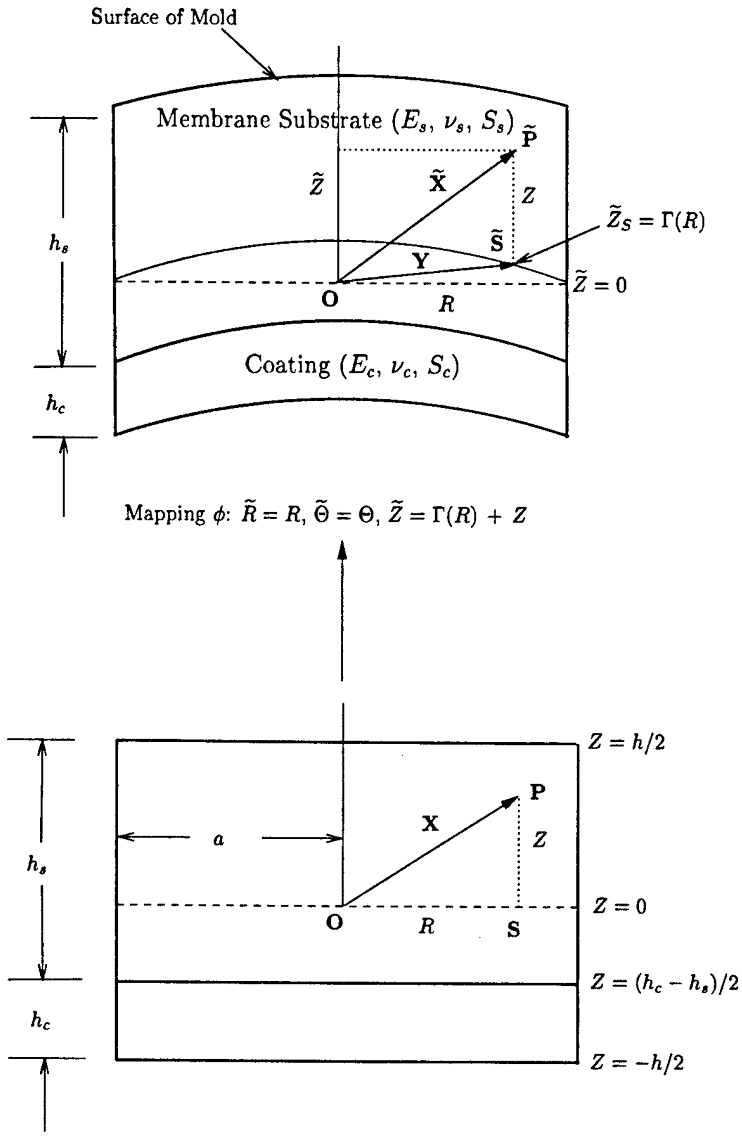

Figure 1: Definition of the reference configuration $\mathcal{S}$ (upper part of Figure) of a coated membrane shell of revolution as a mapping from the reference placement $\mathcal{C}$ (lower part of Figure), assuming the layer thicknesses to be constant along any line parallel to the axis.

where, in (1.1), S_i is the residual stress load in layer i. The elastic constants of the material in each layer appear in the forms

$$Q_i \equiv \frac{E_i}{1-\nu_i^2}, \qquad G_i \equiv \frac{E_i}{1+\nu_i}, \tag{1.8}$$

where E_i and ν_i are Young's modulus and Poisson's ratio for material i. The *areal* density of the coated membrane (mass per unit area of the circular disk perpendicular to the axis; units of kg/m^2) is defined and

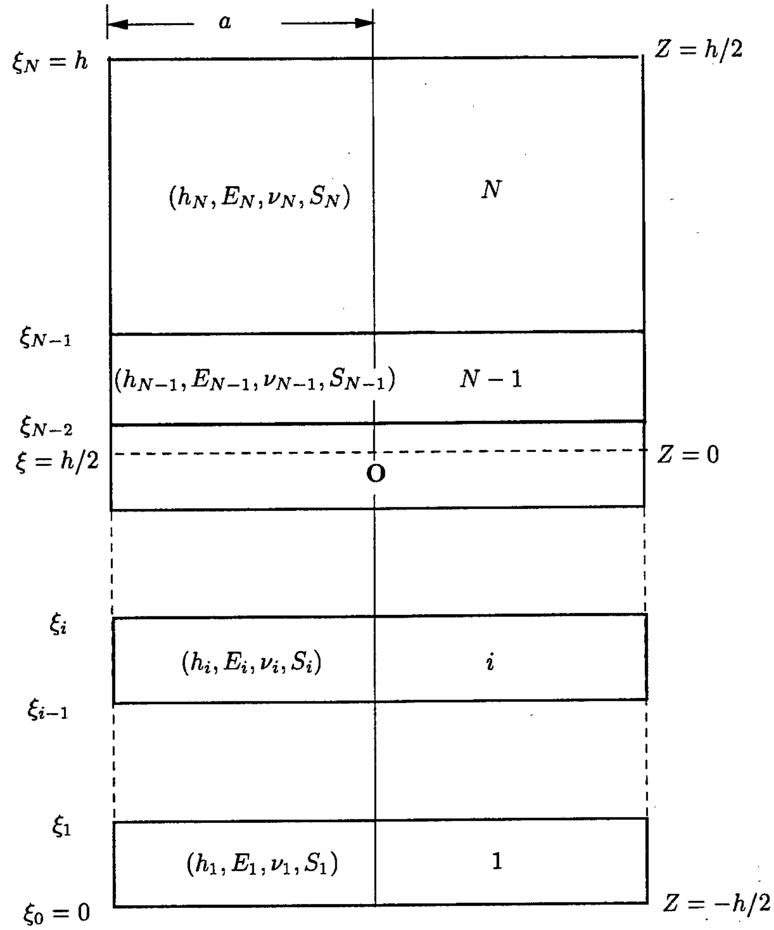

Figure 2: Reference placement $\mathcal{C}$ of an N-layer stack.

denoted by

$$\gamma_0 \equiv \sum_{i=1}^{N} h_i \, \rho_{0i}, \tag{1.9}$$

where ρ_{0i} is the volume mass density of the material in layer i. The external loads are the pressure difference $p = p^- - p^+$ between the two faces of the coated membrane, and the gravitational body force $\gamma_0 \, g$ (where g is the gravitational acceleration) in the positive axial direction (up) of our two Figures.

We realized only after the publication of Ref. [1] that the nine A, B, and D coefficients were not independent. We in fact find the following three relations between the multilayer coefficients (recall that each layer is assumed to be uniform, homogeneous, and isotropic in its material properties):

$$A_\Theta = A - A_\nu, \qquad B_\Theta = B - B_\nu, \qquad D_\Theta = D - D_\nu. \tag{1.10}$$

Similar to Jones [2, p. 155], we refer to A as the *extensional stiffness*, B as the *coupling stiffness*, and D as the *bending stiffness* of the multilayer laminate. It is convenient to also introduce extensional, coupling, and bending Poisson's ratios defined by

$$\nu_A = \frac{A_\nu}{A}, \qquad \nu_B = \frac{B_\nu}{B}, \quad \text{and} \quad \nu_D = \frac{D_\nu}{D}, \tag{1.11}$$

respectively. Note that for a single layer with Poisson's ratio ν, or for a multilayer in which all layers have the *same* Poisson's ratio ν (an unlikely possibility), the three Poisson's ratios defined in (1.11) are all equal to ν as well. Equations (1.10) can be written in terms of the stiffnesses and Poisson's ratios as

$$A_\Theta = A\,(1 - \nu_A), \qquad B_\Theta = B\,(1 - \nu_B), \qquad D_\Theta = D\,(1 - \nu_D). \tag{1.12}$$

The most important initial shape for optical purposes is a paraboloid, so throughout this Report we consider a membrane cast on a paraboloidal mandrel (also referred to as "the mold"), then coated on the mandrel, and subsequently released from the mandrel as a near net-shape coated membrane laminate. We assume that the coated membrane middle surface is governed by the equation

$$\Gamma(R) = \frac{1}{4f}\left(a^2 - R^2\right), \tag{1.13}$$

where f is the focal length of the reference paraboloid. A *flat* middle surface is modeled by a paraboloid having an "infinite" focal length, in which case $\Gamma(R) = 0$. From (1.13), the central (or vertex, or apex) displacement of the paraboloid is given by

$$\Gamma_0 \equiv \Gamma(0) = \frac{a^2}{4f}, \tag{1.14}$$

and its slope at any point R is

$$\Gamma_{,R} = -\frac{R}{2f}. \tag{1.15}$$

The f-number of the paraboloid, which we denote by $F^\#$, is defined by

$$F^\# \equiv \frac{f}{2a}, \tag{1.16}$$

hence the ratio of apex deflection Γ_0 to the radius a can be written as

$$\frac{\Gamma_0}{a} = \frac{1}{8F^\#}. \tag{1.17}$$

In the optics literature a paraboloid with (for example) an f-number of 2 is referred to as an $f/2$ paraboloid, and we indicate this by writing $F^\# = 2$. The ratio (1.17) for an $f/2$ paraboloid is 1/16, or 0.0625.

It is often more convenient to write (1.13) in terms of a parameter κ defined by

$$\kappa \equiv \frac{1}{2f} = \frac{1}{4aF^\#}, \tag{1.18}$$

in which case we have

$$\Gamma(R) = \frac{\kappa}{2}\left(a^2 - R^2\right). \tag{1.19}$$

Note that the principal curvatures κ_1 and κ_2 of the paraboloid are given by

$$\kappa_1 = \frac{-\kappa}{(1 - \kappa^2 R^2)^{3/2}}, \quad \text{and} \quad \kappa_2 = \frac{-\kappa}{(1 - \kappa^2 R^2)^{1/2}}, \tag{1.20}$$

which are equal only at the vertex $R = 0$ of the paraboloid, where $\kappa_1 = \kappa_2 = -\kappa$. Thus, κ is the (negative) vertex curvature of the paraboloid, and $\kappa \to 0$ as $f \to \infty$, i.e., as the paraboloid flattens into a plane.

Unless otherwise stated, all variables appearing in this work are leading order functions of the cylindrical coordinates (R, Θ, Z) on the reference placement $\mathcal{C}$. The subscript (0) used to denote leading order variables in Reference [1] will be omitted in all that follows. Reference [1] should be consulted for further details on the definitions of the functions appearing in the equations of this Report.

2 Geometrically Linear Membrane Laminate Problems

The simplest of the theories presented in Reference [1] is that of a geometrically linear coated membrane laminate. It is the only theory for which we consider solutions that are non-axisymmetric. The equations governing the mechanics of such a laminate are given in §11 of that Report, which are repeated here, beginning with the strain-displacement relations, then the integrated through-the-thickness forms of the constitutive relations, and finally the equilibrium equations stated in terms of stress resultants (stress couples do not appear in a membrane theory). Throughout this Report, a reference to an equation in [1] will be distinguished by appending a ".I" extension to the referenced equation number.

2.1 Pressurized Membrane Laminate with Θ-Dependent Boundary

For membrane laminates, the leading order displacement components are given by

$$U_R = u, \qquad U_\Theta = v, \qquad U_Z = w, \tag{2.1}$$

where u, v, and w are functions of R and Θ only. The leading order strain-displacement relations are found in equations (11.20.I)–(11.22.I), viz.,

$$\epsilon^0_{R\Theta} = \frac{1}{2}\left(v_{,R} + \frac{u_{,\Theta} - v}{R}\right), \tag{2.2}$$

$$\epsilon^0_{RR} = u_{,R}, \tag{2.3}$$

$$\epsilon^0_{\Theta\Theta} = \frac{v_{,\Theta} + u}{R}. \tag{2.4}$$

The stress resultants (11.29.I)–(11.31.I) result from through-the-thickness integrals of the leading order constitutive relations (11.17.I)–(11.19.I), yielding

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta}, \tag{2.5}$$

$$N_{R\Theta} = A_\Theta\,\epsilon^0_{R\Theta}, \tag{2.6}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta}. \tag{2.7}$$

Finally, recalling the definition $\omega(R) = \Gamma(R) + w(R,\Theta)$, the leading order equilibrium equations are given in terms of the stress resultants by equations (11.26.I)–(11.28.I):

$$(RN_R)_{,R} - N_\Theta + N_{R\Theta,\Theta} = 0, \tag{2.8}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R\,N_{\Theta,\Theta} = 0, \tag{2.9}$$

$$\left[R\left(\Gamma_{,R}\,N_R + w_{,R}\,N_R + \frac{w_{,\Theta}}{R}N_{R\Theta}\right)\right]_{,R} + \left(\Gamma_{,R}N_{R\Theta} + w_{,R}N_{R\Theta} + \frac{w_{,\Theta}}{R}N_\Theta\right)_{,\Theta} + (p + \gamma_0\,g)\,R = 0. \tag{2.10}$$

We seek solutions of (2.2)–(2.10) for an initally flat coated membrane, so that $\Gamma_{,R} = 0$, and with $u = v = 0$ for all R. When these hold, we have $\epsilon^0_{R\Theta} = \epsilon^0_{RR} = \epsilon^0_{\Theta\Theta} = 0$, hence $N_R = N_\Theta = \mathcal{N}$ and $N_{R\Theta} = 0$. Equations (2.8) and (2.9) are then identically satisfied, and (2.10) reduces to

$$(R\,w_{,R}\,\mathcal{N})_{,R} + \left(\frac{w_{,\Theta}}{R}\mathcal{N}\right)_{,\Theta} + (p + \gamma_0\,g)\,R = 0. \tag{2.11}$$

Carrying out the differentiations in (2.11), we obtain

$$w_{,RR} + \frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2} = -\frac{(p + \gamma_0 g)}{\mathcal{N}}, \tag{2.12}$$

noting that $\mathcal{N}$ is constant. The left-hand side of this equation is recognized as the Laplacian of w in plane polar coordinates, hence the equation can be written as

$$\nabla^2 w = -\frac{1}{f_0}, \qquad f_0 \equiv \frac{\mathcal{N}}{p + \gamma_0 g}, \tag{2.13}$$

where f_0 is a constant with units of length (m).

We seek solutions of (2.13) in the form of a Fourier series:

$$w(R, \Theta) = \sum_{n=0}^{\infty} \left[A_n(R) \cos(n\Theta) + B_n(R) \sin(n\Theta)\right], \tag{2.14}$$

where the functions $A_n(R)$ and $B_n(R)$ are to be determined. Taking the required partial derivatives of (2.14) and substituting the results in the left-hand side of (2.13), we obtain

$$\sum_{n=0}^{\infty} \Big\{ \left[A_n''(R) \cos(n\Theta) + B_n''(R) \sin(n\Theta)\right] + \frac{1}{R}\left[A_n'(R) \cos(n\Theta) + B_n'(R) \sin(n\Theta)\right] - \frac{n^2}{R^2}\left[A_n(R) \cos(n\Theta) + B_n(R) \sin(n\Theta)\right] \Big\} = -\frac{1}{f_0}, \tag{2.15}$$

where a "prime" denotes a derivative with respect to the argument of a function. Since the trigonometric functions $\sin(n\Theta)$ and $\cos(n\Theta)$ are linearly independent for each value of n, the last equation yields the following ordinary differential equations for $A_n(R)$ and $B_n(R)$:

$$A_0''(R) + \frac{1}{R} A_0'(R) = -\frac{1}{f_0}, \qquad n = 0, \tag{2.16}$$

$$A_n''(R) + \frac{1}{R} A_n'(R) - \frac{n^2}{R^2} A_n(R) = 0, \qquad n \neq 0, \tag{2.17}$$

$$B_n''(R) + \frac{1}{R} B_n'(R) - \frac{n^2}{R^2} B_n(R) = 0, \qquad n \neq 0. \tag{2.18}$$

Equation (2.16) is easily solved to obtain

$$A_0(R) = a_0 - \frac{1}{4f_0} R^2, \tag{2.19}$$

where a_0 is an arbitrary integration constant, and the other integration constant was set equal to zero to insure regularity of $A_0(R)$ at $R = 0$. Equations (2.17) and (2.18) are "equidimensional" [3, §1.6] linear differential equations. They can be reduced to linear differential equations with *constant coefficients* by introducing a new independent variable ξ defined by

$$\xi \equiv \ln R \quad \Rightarrow \quad R = e^{\xi}. \tag{2.20}$$

Setting $A_n(R) = A_n(e^{\xi}) \equiv \widetilde{A}_n(\xi)$ and $B_n(R) = B_n(e^{\xi}) \equiv \widetilde{B}_n(\xi)$, and using the chain rule to compute the derivatives, equation (2.17) reduces to

$$\widetilde{A}_n''(\xi) - n^2 \widetilde{A}_n(\xi) = 0, \qquad n \neq 0, \tag{2.21}$$

and similarly for (2.18). The two linearly independent solutions of (2.21) are $\widetilde{A}_n(\xi) = \mathrm{e}^{n\xi}$ and $\widetilde{A}_n(\xi) = \mathrm{e}^{-n\xi}$, hence $A_n(R) = R^n$ and $A_n(R) = R^{-n}$, so the general solutions of (2.17) and (2.18) are linear combinations of these, viz.,

$$A_n(R) = a_n R^n + c_n R^{-n}, \qquad \text{and} \qquad B_n(R) = b_n R^n + d_n R^{-n}, \tag{2.22}$$

where a_n, b_n, c_n, and d_n are arbitrary constant coefficients depending on n. However, since the solution must be regular at $R = 0$ for any $n \neq 0$, we must set $c_n = d_n = 0$ for each n. Substituting the results into (2.14) yields the general solution:

$$w(R, \Theta) = a_0 - \frac{1}{4f_0} R^2 + \sum_{n=1}^{\infty} \left[a_n R^n \cos(n\Theta) + b_n R^n \sin(n\Theta) \right], \tag{2.23}$$

where the coefficients a_0, a_n and b_n must be determined from the boundary conditions. Typically, one is given the form of the displacement field on the boundary edge $R = a$, i.e., $w(a, \Theta) \equiv W(\Theta)$. The Fourier series coefficients $C_0 \equiv a_0 - a^2/(4f_0)$, $C_n \equiv a_n a^n$, and $D_n \equiv b_n a^n$ are determined in the usual way [3, §5.11] from

$$\begin{aligned} w(a, \Theta) \equiv W(\Theta) &= C_0 + \sum_{n=1}^{\infty} \left[C_n \cos(n\Theta) + D_n \sin(n\Theta) \right], \\ C_0 &= \frac{1}{2\pi} \int_0^{2\pi} W(\Theta)\, d\Theta, \\ C_n &= \frac{1}{\pi} \int_0^{2\pi} W(\Theta) \cos(n\Theta)\, d\Theta, \\ D_n &= \frac{1}{\pi} \int_0^{2\pi} W(\Theta) \sin(n\Theta)\, d\Theta. \end{aligned} \tag{2.24}$$

Having determined C_n and D_n, we can replace $a_n = C_n/a^n$ and $b_n = D_n/a^n$ in (2.23), and the solution takes the final form

$$w(R, \Theta) = C_0 + \frac{1}{4f_0} \left(a^2 - R^2 \right) + \sum_{n=1}^{\infty} \left[C_n (R/a)^n \cos(n\Theta) + D_n (R/a)^n \sin(n\Theta) \right]. \tag{2.25}$$

2.2 Vibrations of a Membrane Laminate

We conclude this brief Section with a digression on the subject of membrane dynamics. We begin by asserting, without proof, that the dynamical leading order *momentum balance* equations are the same as the equilibrium equations (2.8)–(2.10), with the exception that the right-hand sides are replaced by inertia terms comprised of products of the areal mass density γ_0 and an acceleration component (each equation has also been premultiplied by R), i.e.,

$$(RN_R)_{,R} - N_\Theta + N_{R\Theta,\Theta} = \gamma_0\, R\, u_{,tt}, \tag{2.26}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R\, N_{\Theta,\Theta} = \gamma_0\, R\, v_{,tt}, \tag{2.27}$$

$$\begin{aligned} &\left[R \left(\Gamma_{,R} N_R + w_{,R} N_R + \frac{w_{,\Theta}}{R} N_{R\Theta} \right) \right]_{,R} + \left(\Gamma_{,R} N_{R\Theta} + w_{,R} N_{R\Theta} + \frac{w_{,\Theta}}{R} N_\Theta \right)_{,\Theta} \\ &\qquad + (p + \gamma_0\, g)\, R = \gamma_0\, R\, w_{,tt}. \end{aligned} \tag{2.28}$$

We consider free vibration solutions of (2.26)–(2.28) for which $u = v = 0$, assuming there is no pressure load, and ignoring the effects of gravity. We further assume that we have an initally flat reference configuration, so that $\Gamma_{,R} = 0$. The conditions $u = v = 0$ again imply $\epsilon^0_{R\Theta} = \epsilon^0_{RR} = \epsilon^0_{\Theta\Theta} = 0$, hence $N_R = N_\Theta = \mathcal{N}$ and $N_{R\Theta} = 0$. Equations (2.26) and (2.27) are then identically satisfied, and (2.28) reduces to

$$\left(R w_{,R} \mathcal{N}\right)_{,R} + \left(\frac{w_{,\Theta}}{R} \mathcal{N}\right)_{,\Theta} = \gamma_0 R w_{,tt}. \tag{2.29}$$

Assuming $\mathcal{N}$ to be constant, we carry out the differentiations in (2.29) to obtain

$$w_{,RR} + \frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2} = \frac{\gamma_0}{\mathcal{N}} w_{,tt}. \tag{2.30}$$

The left-hand side of (2.30) is the Laplacian of w, hence we can write the equation as

$$\nabla^2 w - \frac{1}{c^2} w_{,tt} = 0, \tag{2.31}$$

which is the homogeneous wave equation for an axial displacement having propagation velocity

$$c = \sqrt{\frac{\mathcal{N}}{\gamma_0}}. \tag{2.32}$$

The solutions of (2.31) are well-known (see, for example, [4, p. 635]), and will not be repeated here. However, the angular frequencies of vibration are given by

$$\omega_{mn} = \alpha_{mn} \frac{c}{a} = \alpha_{mn} \sqrt{\frac{\mathcal{N}}{\gamma_0 a^2}}, \tag{2.33}$$

where a is the membrane radius, and α_{mn} is the mth positive zero of the ordinary Bessel function J_n of order n ($n = 0, 1, 2, \ldots$), i.e., α_{mn} is a solution of the equation $J_n(\alpha_{mn}) = 0$. Note that the tension T which appears in the small oscillation theory of a single material "drumhead" is replaced in this theory of a coated membrane laminate by the net residual stress resultant $\mathcal{N}$ defined in equation (1.1).

3 Geometrically Nonlinear Membrane Laminate Problems

The leading order strain-displacement relations are given by equations (9.27.I)–(9.29.I), which contain nonlinear terms in the axial displacement derivatives:

$$\epsilon^0_{R\Theta} = \frac{1}{2}\left(v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} + \frac{w_{,R} w_{,\Theta}}{R}\right), \tag{3.1}$$

$$\epsilon^0_{RR} = u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2}\left(w_{,R}\right)^2, \tag{3.2}$$

$$\epsilon^0_{\Theta\Theta} = \frac{v_{,\Theta} + u}{R} + \frac{\left(w_{,\Theta}\right)^2}{2R^2}. \tag{3.3}$$

The stress resultants and equilibrium equations are given by equations (9.33.I)–(9.38.I), which are the same as those of the previous §2:

$$N_R = \mathcal{N} + A \epsilon^0_{RR} + A_\nu \epsilon^0_{\Theta\Theta}, \tag{3.4}$$

$$N_{R\Theta} = A_\Theta\, \epsilon^0_{R\Theta}, \tag{3.5}$$

$$N_\Theta = \mathcal{N} + A_\nu\, \epsilon^0_{RR} + A\, \epsilon^0_{\Theta\Theta}. \tag{3.6}$$

and

$$(RN_R)_{,R} - N_\Theta + N_{R\Theta,\Theta} = 0, \tag{3.7}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R\, N_{\Theta,\Theta} = 0, \tag{3.8}$$

$$\left[R\left(\Gamma_{,R}\, N_R + w_{,R}\, N_R + \frac{w_{,\Theta}}{R} N_{R\Theta}\right)\right]_{,R} + \left(\Gamma_{,R} N_{R\Theta} + w_{,R} N_{R\Theta} + \frac{w_{,\Theta}}{R} N_\Theta\right)_{,\Theta} + (p + \gamma_0\, g)\, R = 0. \tag{3.9}$$

3.1 Reduction to an Axisymmetric System

We specialize immediately to the case of an *axisymmetric* system by assuming that none of the variables depend on the angular coordinate Θ. Thus, all terms involving partial derivatives with respect to Θ vanish, leaving the following simplified set of strain-displacement relations and equilibrium equations:

$$\epsilon^0_{R\Theta} = \frac{1}{2}\left(v_{,R} - \frac{v}{R}\right), \tag{3.10}$$

$$\epsilon^0_{RR} = u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2}\left(w_{,R}\right)^2, \tag{3.11}$$

$$\epsilon^0_{\Theta\Theta} = \frac{u}{R}, \tag{3.12}$$

$$(RN_R)_{,R} - N_\Theta = 0, \tag{3.13}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} = 0, \tag{3.14}$$

$$\left[R\left(\Gamma_{,R}\, N_R + w_{,R}\, N_R\right)\right]_{,R} + (p + \gamma_0\, g)\, R = 0. \tag{3.15}$$

Equations (3.14) and (3.15) can be integrated once to obtain

$$R^2 N_{R\Theta} = C_{R\Theta}, \tag{3.16}$$

and

$$R\left(\Gamma_{,R}\, N_R + w_{,R}\, N_R\right) + (p + \gamma_0\, g)\, \frac{R^2}{2} = C_R, \tag{3.17}$$

where $C_{R\Theta}$ and C_R are arbitrary integration constants. We must set both of these constants equal to zero to insure that both $N_{R\Theta}$ and N_R are regular at the origin $R = 0$. Thus, (3.16) and (3.17) reduce to

$$N_{R\Theta} = 0, \text{ for all } R. \tag{3.18}$$

$$\Gamma_{,R}\, N_R + w_{,R}\, N_R + (p + \gamma_0\, g)\, \frac{R}{2} = 0. \tag{3.19}$$

From (3.18), (3.5) and (3.10) it then follows that

$$\epsilon^0_{R\Theta} = \frac{1}{2}\left(v_{,R} - \frac{v}{R}\right) = 0, \tag{3.20}$$

since A_Θ is in general nonzero. Equation (3.20) is easily solved to obtain $v(R) = v_0 R$, where v_0 is an arbitrary constant. We observe that if v vanishes for *any* non-zero value of R, say $R_0 \neq 0$, then $v(R_0) = v_0 R_0 = 0$, hence $v_0 = 0$, and so v vanishes for *all* R.

The system of equations that remains to be solved is then

$$\epsilon^0_{RR} = u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2}(w_{,R})^2, \qquad \epsilon^0_{\Theta\Theta} = \frac{u}{R}, \tag{3.21}$$

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta}, \tag{3.22}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta}, \tag{3.23}$$

$$(RN_R)_{,R} - N_\Theta = 0, \tag{3.24}$$

$$\Gamma_{,R} N_R + w_{,R} N_R + (p + \gamma_0\, g)\,\frac{R}{2} = 0. \tag{3.25}$$

3.2 Generalization of Hencky-Campbell Theory to a Membrane Laminate

We continue the reduction by specializing now to an initially flat coated membrane, so that $\Gamma_{,R} = 0$, reducing the system of equations to

$$\epsilon^0_{RR} = u_{,R} + \frac{1}{2}(w_{,R})^2, \qquad \epsilon^0_{\Theta\Theta} = \frac{u}{R}, \tag{3.26}$$

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta}, \tag{3.27}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta}, \tag{3.28}$$

$$(RN_R)_{,R} - N_\Theta = 0 \quad \Rightarrow \quad N_\Theta = N_R + R\,N_{R,R}, \tag{3.29}$$

$$w_{,R} N_R + (p + \gamma_0\, g)\,\frac{R}{2} = 0. \tag{3.30}$$

This system is augmented by the usual compatibility condition obtained by eliminating u between the two strain-displacement relations (3.26), yielding (see, for example, [5]):

$$R\,\epsilon^0_{\Theta\Theta,R} = \epsilon^0_{RR} - \epsilon^0_{\Theta\Theta} - \frac{1}{2}(w_{,R})^2. \tag{3.31}$$

We proceed by introducing a dimensionless coordinate ρ and dimensionless displacement components $\widehat{u}$ and $\widehat{w}$ defined by

$$\rho \equiv \frac{R}{a}, \qquad \widehat{u} \equiv \frac{u}{a}, \qquad \widehat{w} \equiv \frac{w}{a}, \tag{3.32}$$

together with the following dimensionless constants:

$$\nu_A \equiv \frac{A_\nu}{A}, \qquad \overline{E} \equiv \frac{A}{h}\left(1-\nu_A^2\right), \qquad \tau \equiv \frac{\mathcal{N}}{\overline{E}\,h}, \qquad q \equiv \frac{(p+\gamma_0 g)\,a}{\overline{E}\,h}, \tag{3.33}$$

and two new dimensionless dependent variables x_r and x_θ defined by

$$x_r \equiv \frac{N_R - \mathcal{N}}{\overline{E}\,h}, \qquad x_\theta \equiv \frac{N_\Theta - \mathcal{N}}{\overline{E}\,h}. \tag{3.34}$$

Note that ν_A is the extensional Poisson's ratio defined earlier in (1.11), while $\overline{E}$ acts as the "effective" Young's modulus of the coated membrane laminate. Equations (3.26)–(3.31) can be rewritten in terms of these dimensionless quantities as

$$\epsilon^0_{RR} = \widehat{u}_{,\rho} + \frac{1}{2}\widehat{w}^2_{,\rho}, \qquad \epsilon^0_{\Theta\Theta} = \frac{\widehat{u}}{\rho}, \tag{3.35}$$

$$x_r = \left(\frac{1}{1-\nu_A^2}\right)\left(\epsilon^0_{RR} + \nu_A\,\epsilon^0_{\Theta\Theta}\right), \tag{3.36}$$

$$x_\theta = \left(\frac{1}{1-\nu_A^2}\right)\left(\epsilon^0_{\Theta\Theta} + \nu_A\,\epsilon^0_{RR}\right), \tag{3.37}$$

$$x_\theta = x_r + \rho\,x_{r,\rho} \equiv \left(\rho\,x_r\right)_{,\rho}, \tag{3.38}$$

$$\widehat{w}_{,\rho}\left(\tau + x_r\right) + q\,\frac{\rho}{2} = 0. \tag{3.39}$$

$$\rho\,\epsilon^0_{\Theta\Theta,\rho} = \epsilon^0_{RR} - \epsilon^0_{\Theta\Theta} - \frac{1}{2}\widehat{w}^2_{,\rho}, \tag{3.40}$$

These equations have precisely the same *form* as Campbell's [6] equations, which are themselves modifications of Hencky's [7] membrane equations to allow for an initial tension in the pressurized membrane. Equations (3.36) and (3.37) are easily inverted to obtain

$$\epsilon^0_{RR} = x_r - \nu_A\,x_\theta, \tag{3.41}$$

$$\epsilon^0_{\Theta\Theta} = x_\theta - \nu_A\,x_r, \tag{3.42}$$

which, together with (3.38), can be substituted in (3.40) to get the compatibility condition in terms of x_r and x_θ:

$$\rho\left(x_r + x_\theta\right)_{,\rho} = -\frac{1}{2}\widehat{w}^2_{,\rho}. \tag{3.43}$$

We now use Equation (3.39) to eliminate $\widehat{w}_{,\rho}$ in equation (3.43), yielding the following equation involving only the x's:

$$\left(\tau + x_r\right)^2\left[\frac{\left(x_r + x_\theta\right)_{,\rho}}{\rho}\right] + \frac{q^2}{8} = 0. \tag{3.44}$$

It is convenient here to rescale τ, x_r, x_θ, and $\widehat{w}$, i.e., we set

$$\tau = \frac{1}{4}q^{2/3}\,\overline{\tau}, \qquad x_r = \frac{1}{4}q^{2/3}\,\overline{x}_r, \qquad x_\theta = \frac{1}{4}q^{2/3}\,\overline{x}_\theta, \qquad \widehat{w} = q^{1/3}\,\overline{w} \tag{3.45}$$

to write equations (3.39) and (3.44) as

$$\overline{w}_{,\rho}\left(\overline{\tau} + \overline{x}_r\right) + 2\rho = 0, \tag{3.46}$$

and

$$(\overline{\tau} + \overline{x}_r)^2 \left[\frac{(\overline{x}_r + \overline{x}_\theta)_{,\rho}}{\rho}\right] + 8 = 0, \tag{3.47}$$

respectively.

We assume a power series solution for $\overline{x}_r(\rho)$, and find after some numerical experimentation that only even powers of ρ will contribute. It is convenient to write the series in the form

$$\overline{x}_r = -\overline{\tau} + b_0 \left(1 + \sum_{n=1}^{\infty} b_{2n}\, \rho^{2n}\right). \tag{3.48}$$

From this and the second equality of (3.38) it follows that

$$\overline{x}_\theta = (\rho\, x_r)_{,\rho} = -\overline{\tau} + b_0 \left[1 + \sum_{n=1}^{\infty} (2n+1)\, b_{2n}\, \rho^{2n}\right]. \tag{3.49}$$

The sum of the last two equations yields

$$\overline{x}_r + \overline{x}_\theta = -2\,\overline{\tau} + 2b_0 \left[1 + \sum_{n=1}^{\infty} (n+1)\, b_{2n}\, \rho^{2n}\right], \tag{3.50}$$

hence, noting that $\overline{\tau}$ is a constant,

$$\frac{(\overline{x}_r + \overline{x}_\theta)_{,\rho}}{\rho} = 4\, b_0 \sum_{n=1}^{\infty} n\,(n+1)\, b_{2n}\, \rho^{2n-2}. \tag{3.51}$$

This expression, together with (3.48), is now substituted in (3.47), and the coefficients b_{2n}, $n \geq 1$, are then determined by equating to zero the coefficients of like powers of ρ. We find, using the *Mathematica* computer algebra system, that b_{2n} is given in terms of b_0 by

$$b_{2n} = -\frac{\beta_{2n}}{b_0^{3n}}, \qquad n \geq 1, \tag{3.52}$$

where the β_{2n} are purely numerical coefficients given, for $1 \leq n \leq 9$, by

$$\begin{gathered}
\beta_2 = 1, \qquad \beta_4 = \frac{2}{3}, \qquad \beta_6 = \frac{13}{18}, \qquad \beta_8 = \frac{17}{18}, \qquad \beta_{10} = \frac{37}{27}, \\
\beta_{12} = \frac{1205}{567}, \qquad \beta_{14} = \frac{219241}{63504}, \qquad \beta_{16} = \frac{6634069}{1143072}, \qquad \beta_{18} = \frac{51523763}{5143824}.
\end{gathered} \tag{3.53}$$

In order to determine the coefficient b_0 (in terms of which all the other coefficients b_{2n} can be calculated), we must impose boundary conditions.

We consider here a clamped boundary, requiring $u(a) = 0$ and $w(a) = 0$, or equivalently, $\widehat{u}(1) = 0$ and $\widehat{w}(1) = 0$. From the second equation of (3.35), together with equation (3.42), we have the following power series representation for $\widehat{u}$:

$$\widehat{u}(\rho) = \rho\, \epsilon^0_{\Theta\Theta} = \rho\, (x_\theta - \nu_A\, x_r) = \frac{1}{4} q^{2/3} \rho \left[-\overline{\tau}\,(1-\nu_A) + b_0\,(1-\nu_A) + b_0 \sum_{n=1}^{\infty} (2n+1-\nu_A)\, b_{2n} \rho^{2n}\right]. \tag{3.54}$$

Applying to this expression the boundary condition on $\widehat{u}$ at $\rho = 1$ yields an equation which must be solved numerically for b_0:

$$b_0 \left[1 - \nu_A + \sum_{n=1}^{\infty} (2n+1-\nu_A)\, b_{2n}\right] - (1-\nu_A)\,\overline{\tau} = 0. \tag{3.55}$$

After determining b_0, which in general can be seen from (3.55) to depend on both ν_A and $\overline{\tau}$, the series solution for the radial displacement $\widehat{u}(\rho)$ is given by (3.54).

The dimensionless axial component of displacement $\overline{w}(\rho)$ can be obtained by assuming an even power series of the form

$$\overline{w}(\rho) = \sum_{n=0}^{\infty} c_{2n}\,\rho^{2n}, \tag{3.56}$$

the derivative of which is

$$\overline{w}_{,\rho} = 2\sum_{n=0}^{\infty} n\,c_{2n}\,\rho^{2n-1}. \tag{3.57}$$

This is substituted, together with (3.48) and (3.51), in (3.46), to obtain an equation from which the coefficients c_{2n}, $n \geq 1$, can be determined by comparing like powers of ρ. Again using the Mathematica computer algebra system, we find the coefficients c_{2n}, $n \geq 1$, to be given in terms of b_0 by

$$c_{2n} = -\frac{\gamma_{2n}}{b_0^{3n}}\,b_0^2, \qquad n \geq 1, \tag{3.58}$$

where the γ_{2n} are purely numerical coefficients given, for $1 \leq n \leq 9$, by

$$\begin{gathered}\gamma_2 = 1, \qquad \gamma_4 = \frac{1}{2}, \qquad \gamma_6 = \frac{5}{9}, \qquad \gamma_8 = \frac{55}{72}, \qquad \gamma_{10} = \frac{7}{6}, \\ \gamma_{12} = \frac{205}{108}, \qquad \gamma_{14} = \frac{17051}{5292}, \qquad \gamma_{16} = \frac{2864485}{508032}, \qquad \gamma_{18} = \frac{103863265}{10287648}.\end{gathered} \tag{3.59}$$

The remaining constant c_0 is determined by the clamped edge boundary condition $\overline{w}(1) = 0$, hence from (3.56):

$$c_0 = -\sum_{n=1}^{\infty} c_{2n} = b_0^2 \sum_{n=1}^{\infty} \frac{\gamma_{2n}}{b_0^{3n}}. \tag{3.60}$$

After determining c_0, the dimensionless axial displacement component $\overline{w}(\rho)$ is given by the power series (3.56).

Power series solutions for the displacement components (as well as the stress and strain components) having the proper physical units can be obtained by restoring the scale factors introduced in equations (3.32), (3.34), and (3.45). For example,

$$u(R) = a\,\widehat{u}(R/a) = \frac{1}{4}q^{2/3}R\left[-\overline{\tau}\,(1-\nu_A) + b_0\,(1-\nu_A) + b_0\sum_{n=1}^{\infty}(2n+1-\nu_A)\,b_{2n}(R/a)^{2n}\right], \tag{3.61}$$

and

$$w(R) = a\,\widehat{w}(R/a) = a\,q^{1/3}\,\overline{w}(R/a) = a\,q^{1/3}\sum_{n=0}^{\infty} c_{2n}\,(R/a)^{2n}, \tag{3.62}$$

and the apex deflection $w_0 \equiv w(0)$ is, from (3.62):

$$w_0 = a\,q^{1/3}\,c_0. \tag{3.63}$$

3.3 Applications to Bulge Testing

The last equation (3.63) is useful for the analysis of data obtained from bulge testing (see, for example, [8, 9, 10] for descriptions of such tests), a method under consideration for experimental determination of the intrinsic coating stress. A brief discussion of such an analysis will follow, but first we compare graphs of apex deflection versus pressure, using (3.63), to those obtained from FE analysis. Figure 3 shows the results when $\mathcal{N} = 0$, i.e., for a zero net residual stress resultant. The coated membrane radius here is $a = 3.0$

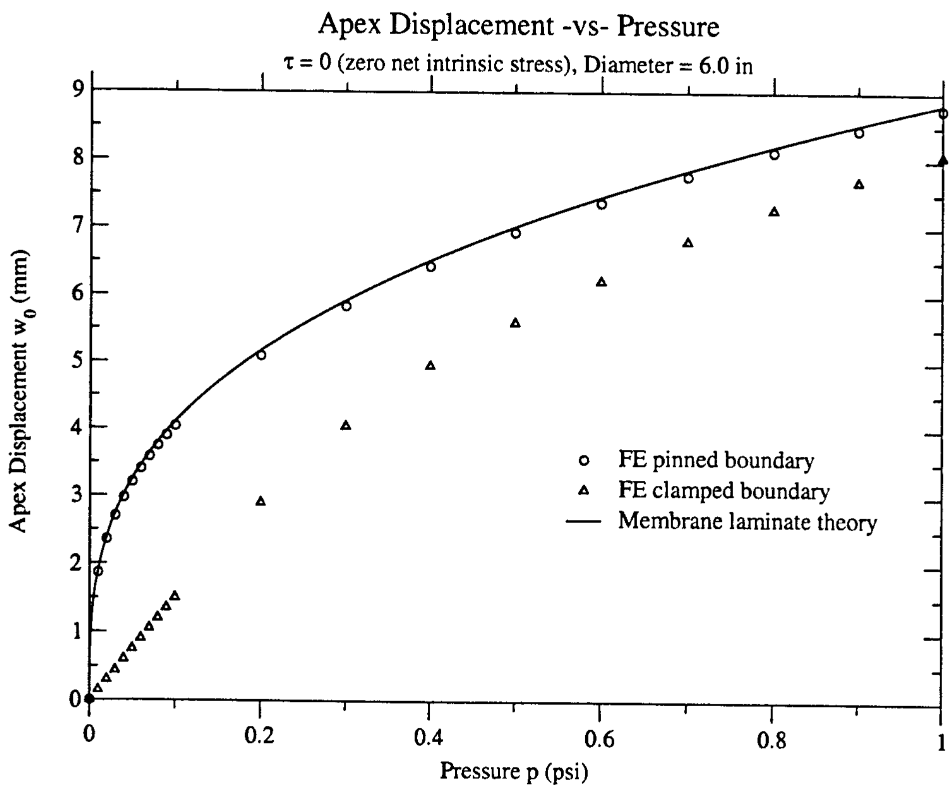

Figure 3: Comparisons of w_0 versus p, using equation (3.63), to results using the finite element method. These plots are for $\mathcal{N} = 0$ (hence also $\tau = \overline{\tau} = 0$), i.e., for zero net residual stress in the coated membrane laminate.

inches = 0.0762 m, the coating and membrane thicknesses are $h_1 = h_c = 1\,\mu$m and $h_2 = h_s = 20\,\mu$m, respectively, with $E_c = 44.0$ GPa and $E_s = 2.2$ GPa. The two Poisson's ratios have been assumed the same, i.e., $\nu_c = \nu_s = 0.4$. FE results for both pinned edge and clamped edge boundary conditions (see §4.2.3 for the definition of pinned edge boundary conditions), treating the coated membrane as a "plate", are also shown in Figure 3. It is clear from these plots that the results of the theory under consideration resemble more closely the FE results for a plate with a pinned boundary rather than one with a clamped boundary. We interpret this to mean that the coated membrane behaves more like a plate than a "true membrane", i.e., the bending stiffness of the coated membrane is large enough in this example that a geometrically nonlinear *plate* theory is probably necessary to obtain agreement with the FE clamped edge results.

Returning to the subject of bulging tests, we recall that in such tests the apex deflection ("bulge") w_0 is measured as the pressure load p is varied. The analysis of data obtained in this way often relies on a formula proposed by Beams [8] (see, for example, [9, p. 254], cited in the recent book by Ohring [10, p. 716]). Beams cites for this formula apparently unpublished work by Cabrera. The derivation of the formula is not given in [8], and it was used there and in [9, 11] only to analyze thin films of a single material, not a laminate. A similar formula was given in [12] for applications to square and rectangular films of a single material. These formulas seem to be intended for applications to "true membranes", since they do not include bending stiffness. The formula given by Beams [8], expressed in our notation (Beams' application was to a *single* thin film of gold), is:

$$p = 4\frac{h}{a}\frac{w_0}{a}\left(T_0 + \frac{2}{3}\frac{E}{1-\nu}\frac{w_0^2}{a^2}\right), \qquad \text{equation (2) of [8]}, \tag{3.64}$$

where T_0 is referred to as the "tension" by Beams, but has units of *stress* and corresponds to our $\mathcal{N}/h$. Since Poisson's ratio for gold is approximately 0.4, for Beams' application equation (3.64) can be rewritten as

$$p = 4\,\frac{w_0}{a^2}\mathcal{N} + 4.44\,E\,h\,\frac{w_0^3}{a^4}, \qquad \mathcal{N} \equiv h\,T_0. \tag{3.65}$$

For a single material, in which case $\overline{E} = E$, the two terms of the Beams/Cabrera formula follow as an *approximation* of our theory by investigating the apex deflection as a function of pressure in the two limiting cases of zero residual stress, and large residual stress. For zero residual stress ($\mathcal{N} = 0$), equation (3.63) applies, where c_0 is a function of Poisson's ratio only, which we find to have the value 0.626 for $\nu = 0.4$, i.e.,

$$w_0 = 0.626\,a\left(\frac{p\,a}{Eh}\right)^{1/3}, \quad \mathcal{N} = 0, \quad \nu = 0.4. \tag{3.66}$$

On the other hand, for large values of $\mathcal{N}$, i.e., of $\overline{\tau}$, the coefficient b_0 in (3.60) is found to be large, so that $c_0 \approx 1/b_0$. In this case it also follows from (3.55) that $b_0 \approx \overline{\tau}$, hence $c_0 \approx 1/\overline{\tau}$ and from (3.63)

$$w_0 \approx a\frac{q^{1/3}}{\overline{\tau}}.$$

But from (3.45), $\overline{\tau} = 4\tau/q^{2/3}$, which yields

$$w_0 \approx \frac{a}{4\tau}q = \frac{a^2}{4\mathcal{N}}\,p, \quad \mathcal{N} \text{ large}, \tag{3.67}$$

where we used (3.45) again to replace q and τ. Thus, for large $\mathcal{N}$, w_0 is linear with pressure p. Equations (3.67) and (3.66) yield two expressions for the pressure as a function of w_0:

$$p_1 \approx 4\,\frac{w_0}{a^2}\,\mathcal{N}, \quad \mathcal{N} \text{ large}, \tag{3.68}$$

and

$$p_2 = 4.08\,Eh\frac{w_0^3}{a^4}, \quad \mathcal{N} = 0, \quad \nu = 0.4, \tag{3.69}$$

respectively. The Beams/Cabrera formula (3.65) is obtained as a simple linear combination of these expressions for p_1 and p_2, viz.,

$$p = p_1 + \frac{4.44}{4.08}\,p_2. \tag{3.70}$$

Since this formula is an approximation, it is expected not to be as accurate in determining the intrinsic stress as one determined from the exact theory. A more systematic derivation of Beams/Cabrera-like formulae using perturbation techniques will be given later in §5.2.

Our algorithm for using the theory to compute the coating stress S_c in a single-layer coating from bulge test data is the following. For a given pressure p, a measurement of the apex displacement w_0 is made. With p we can compute q from (3.33), and then from (3.63) we have the corresponding value of c_0:

$$c_0 = \frac{w_0}{a\,q^{1/3}}. \tag{3.71}$$

For this value of c_0, we now determine b_0 by solving, using some numerical method, equation (3.60), i.e.,

$$c_0 - b_0^2\sum_{n=1}^{\infty}\frac{\gamma_{2n}}{b_0^{3n}} = 0, \tag{3.72}$$

rather than (3.55). This, in effect, gives us b_0 as a function of c_0, i.e., $b_0(c_0)$. Substitution of this value of b_0 in (3.55) then gives $\overline{\tau}$ as a function of ν_A and b_0 (or, equivalently, c_0):

$$\overline{\tau}(c_0, \nu_A) = b_0\left[1 + \frac{1}{(1-\nu_A)}\sum_{n=1}^{\infty}(2n+1-\nu_A)\,b_{2n}\right]. \tag{3.73}$$

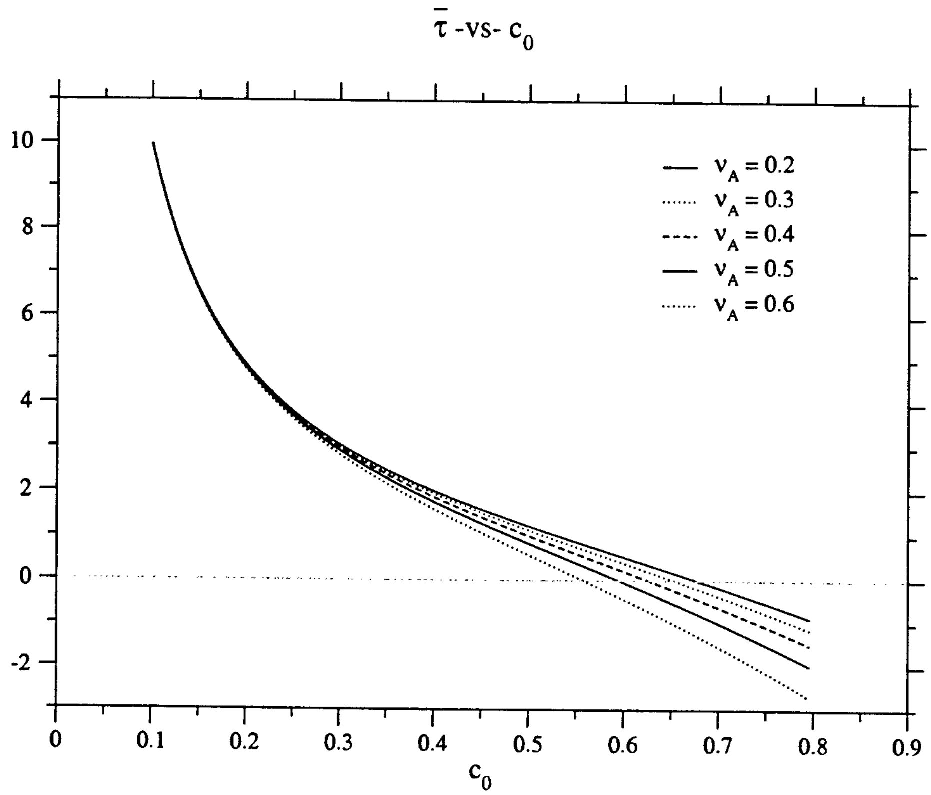

Figure 4: $\overline{\tau}$ versus c_0 for five values of ν_A.

We assume that the geometrical and material properties of the coating and membrane are known, so that ν_A can be calculated from its definition (3.33), and then (3.73) allows us to compute $\overline{\tau}(c_0)$ for this fixed value of ν_A. As the final step, the computed value of $\overline{\tau}$ is used in definitions (3.45) and (3.33) to obtain $\mathcal{N}$, from which S_c follows according to definition (1.1), assuming that the membrane substrate residual stress S_s is known. Figure 4 shows the graphs of $\overline{\tau}(c_0)$ versus c_0 for five different values of ν_A (note that the curve for $\nu_A = 0.3$, restricted to the interval $0.1 \le c_0 \le 0.653$, is the inverse of the graph shown in Figure 6 of Campbell [6]).

In closing this Section, we note that in order to include the bending stiffness of the material, Bonnotte, et al [13] appended an additional term to a Beams/Cabrera-like formula. They then used FE analysis to determine two fitting constants appearing in their formula. This model was applied later by Bahr, et al [14] to a square, five-layer *laminate* material, by simply replacing E and ν in the Bonnotte, et al formula with thickness-weighted "composite" values, i.e.,

$$E = \sum_{i=1}^{5} h_i E_i/h, \qquad \nu = \sum_{i=1}^{5} h_i \nu_i/h, \qquad h = \sum_{i=1}^{5} h_i. \tag{3.74}$$

These composite values appear in their equation (1) only in the combination $E/(1-\nu^2)$, which they refer to as the composite "biaxial modulus", denoted by Y_B in Bonnotte, et al [13], and by Q in our definition (1.8). It should be noted in this regard that in other recent literature, e.g., [15, 16, 17], the term "biaxial modulus" is used instead for the quantity $E/(1-\nu)$ (a convention which we shall adopt later, denoting it by $\mathcal{B}$). For a Poisson's ratio of 0.4 the first definition gives a value of $1.19\,E$, while the second gives a value of $1.67\,E$, roughly 40% higher than the first. In Table 1 we use the Bahr, et al [14] data to compare values

Table 1: Comparison of thickness-weighted averages with values occurring in the geometrically nonlinear membrane laminate theory.

	E (GPa)	ν	$E/(1-\nu^2)$ (GPa)
Thickness-weighted averages	145.6	0.280	158.0
Membrane laminate theory	145.8	0.283	158.5

computed using the thickness-weighted averages in (3.74) with those computed using the definitions (3.33) occurring in the geometrically nonlinear membrane laminate theory. For all practical purposes, the values obtained by the two different methods are the same in this particular example.

Results obtained with the formulas used in [13] and [14] are probably not comparable to results using our theory, as their formulas were determined empirically by fitting to data obtained on *rectangular* specimens, rather than circular ones. However, ignoring this important distinction for the moment, and using the data given in [14], we compare in Figure 5 the results of the theory with the results shown in Figure 1 of [14]. Note that their data is for a multilayer consisting of four coating material layers placed on a (fifth) silicon

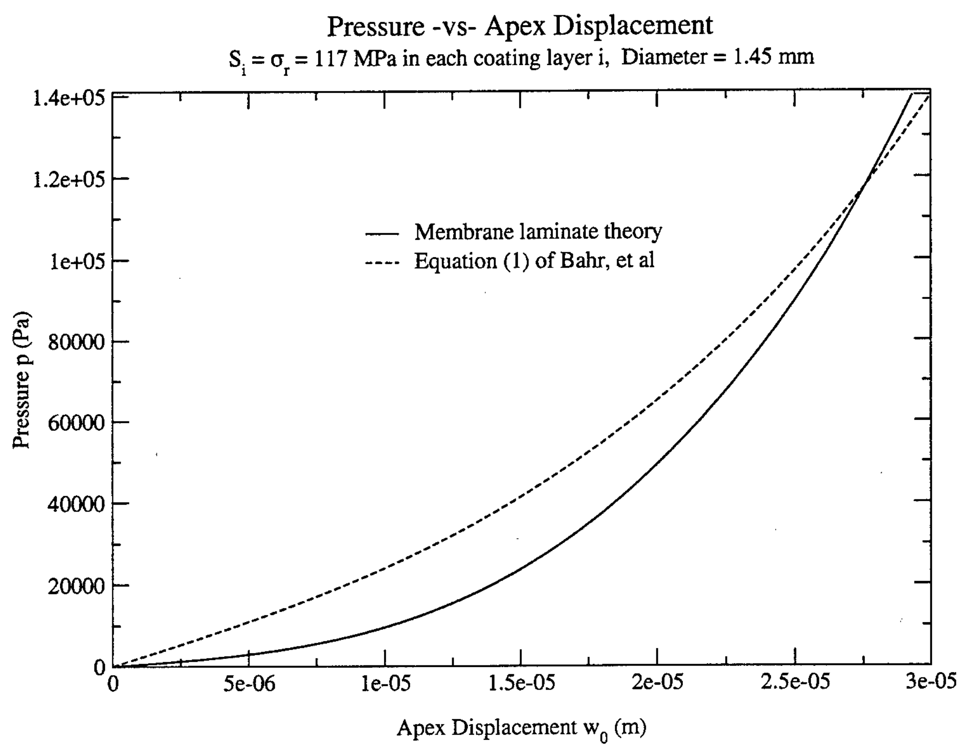

Figure 5: Comparison of membrane laminate theory, equation (3.63), with curve generated by equation (1) of Bahr, et al [14], assuming $S_i = \sigma_r = 117$ MPa in each coating layer i.

substrate layer, so that the constants of our theory, defined in equations (1.1)–(1.7), involve $N = 5$ layers. For the theory, we took the radius to be $a = 0.725$ mm, half that of the side length 1.45 mm of their square specimens (in [14] this side length is also denoted by a, which should not be confused with our radius a). In order to match the final point $(3 \times 10^{-5}\ \mathrm{m}, 1.4 \times 10^5\ \mathrm{Pa})$ of their Figure 1 using their equation (1), we had to take their residual stress σ_r to be 117 MPa, although they mention the range to be from 90 MPa to 110 MPa. Our theoretical curve was generated using equation (3.63), *assuming* that $S_i = 117$ MPa for each layer i of the *four-layer coating*, and $S_5 = 0$ in the substrate. The resulting comparison is reminiscent of our comparison of the theory and FE analysis in Figure 3, i.e., for a given pressure the theory generally overestimates the corresponding apex deflection.

4 Geometrically Linear Shell Laminate Problems

For shell laminates, the leading order displacement components are given by the Kirchhoff-Love expressions

$$U_R = u - Z w_{,R}, \qquad U_\Theta = v - Z \frac{w_{,\Theta}}{R}, \qquad U_Z = w, \tag{4.1}$$

where u, v, and w are functions of R and Θ only. For a geometrically linear shell laminate, the strain components depend linearly on these components and their partial derivatives, according to equations (10.23.I)–(10.28.I) of Reference [1]:

$$\epsilon_{R\Theta} = \frac{1}{2}\left[v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} + 2Z\left(\frac{w_{,\Theta}}{R^2} - \frac{w_{,R\Theta}}{R}\right)\right] \equiv \epsilon^0_{R\Theta} - Z\, k_{R\Theta}, \tag{4.2}$$

$$\epsilon_{RR} = u_{,R} + \Gamma_{,R} w_{,R} - Z\, w_{,RR} \equiv \epsilon^0_{RR} - Z\, k_{RR}, \tag{4.3}$$

$$\epsilon_{\Theta\Theta} = \frac{v_{,\Theta} + u}{R} - Z\left(\frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2}\right) \equiv \epsilon^0_{\Theta\Theta} - Z k_{\Theta\Theta}, \tag{4.4}$$

where the Z-independent strains and curvatures are given by

$$\epsilon^0_{R\Theta} \equiv \frac{1}{2}\left(v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R}\right), \qquad k_{R\Theta} \equiv -\frac{w_{,\Theta}}{R^2} + \frac{w_{,R\Theta}}{R}, \tag{4.5}$$

$$\epsilon^0_{RR} \equiv u_{,R} + \Gamma_{,R} w_{,R}, \qquad k_{RR} \equiv w_{,RR}, \tag{4.6}$$

$$\epsilon^0_{\Theta\Theta} \equiv \frac{v_{,\Theta} + u}{R}, \qquad k_{\Theta\Theta} \equiv \frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2}. \tag{4.7}$$

The stress resultants and couples for a shell laminate are given by equations (10.39.I)–(10.44.I) of Reference [1]:

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}, \tag{4.8}$$

$$N_{R\Theta} = A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta}, \tag{4.9}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta} + B_\nu\,k_{RR} + B\,k_{\Theta\Theta}, \tag{4.10}$$

$$M_R = -\mathcal{M} - B\,\epsilon^0_{RR} - B_\nu\,\epsilon^0_{\Theta\Theta} - D\,k_{RR} - D_\nu\,k_{\Theta\Theta}, \tag{4.11}$$

$$M_{R\Theta} = -B_\Theta\,\epsilon^0_{R\Theta} - D_\Theta\,k_{R\Theta}, \tag{4.12}$$

$$M_\Theta = -\mathcal{M} - B_\nu\,\epsilon^0_{RR} - B\,\epsilon^0_{\Theta\Theta} - D_\nu\,k_{RR} - D\,k_{\Theta\Theta}, \tag{4.13}$$

but note that the Z-independent strains are now replaced by (4.5)–(4.7) of the present Section. The equilibrium equations for a shell laminate are equations (10.36.I)–(10.38.I) of Reference [1], which are similar to those for a membrane, but contain additional terms involving the stress couples:

$$(RN_R)_{,R} - N_\Theta + N_{R\Theta,\Theta} = 0, \tag{4.14}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R N_{\Theta,\Theta} = 0, \tag{4.15}$$

and

$$\left[R\,\Gamma_{,R}\,N_R + (R\,M_R)_{,R} - M_\Theta + M_{R\Theta,\Theta}\right]_{,R} + \left[\Gamma_{,R}N_{R\Theta} + \frac{1}{R^2}\left(R^2\,M_{R\Theta}\right)_{,R} + \frac{1}{R}\,M_{\Theta,\Theta}\right]_{,\Theta} + (p + \gamma_0\,g)\,R = 0. \tag{4.16}$$

4.1 Reduction to an Axisymmetric System

We specialize now to the case of an axisymmetric system by assuming that none of the variables depend on the angular coordinate Θ. Thus, all terms involving partial derivatives with respect to Θ vanish, leaving the following simplified set of strain-displacement relations:

$$\epsilon^0_{R\Theta} = \frac{1}{2}\left(v_{,R} - \frac{v}{R}\right), \qquad k_{R\Theta} = 0, \tag{4.17}$$

$$\epsilon^0_{RR} = u_{,R} + \Gamma_{,R}w_{,R}, \qquad k_{RR} = w_{,RR}, \tag{4.18}$$

$$\epsilon^0_{\Theta\Theta} = \frac{u}{R}, \qquad k_{\Theta\Theta} = \frac{w_{,R}}{R}, \tag{4.19}$$

and equilibrium equations:

$$(RN_R)_{,R} - N_\Theta = 0, \tag{4.20}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} = 0, \tag{4.21}$$

$$\left[R\,\Gamma_{,R}\,N_R + (R\,M_R)_{,R} - M_\Theta\right]_{,R} + (p + \gamma_0\,g)\,R = 0. \tag{4.22}$$

As in §3, *cf.* equations (3.16) and (3.17), equations (4.21) and (4.22) can be integrated immediately. And, in order for the results to be regular at $R = 0$, we must again set the integration constants to zero, which yields the results

$$N_{R\Theta} = 0, \text{ for all } R. \tag{4.23}$$

$$R\,\Gamma_{,R}\,N_R + (R\,M_R)_{,R} - M_\Theta + (p + \gamma_0\,g)\,\frac{R^2}{2} = 0. \tag{4.24}$$

Since $k_{R\Theta} = 0$, we have from (4.9) and (4.23) that $\epsilon^0_{R\Theta} = 0$ (since A_Θ is, in general, nonzero). Thus, as in §3, it follows that $v(R) = v_0\,R$, where v_0 is an arbitrary integration constant; if $v(R_0) = 0$ for any non-zero value R_0 of R, then $v(R) = 0$ for all R. Since both $\epsilon^0_{R\Theta} = 0$ and $k_{R\Theta} = 0$, it follows from (4.12) that

$$M_{R\Theta} = 0, \text{ for all } R. \tag{4.25}$$

The remaining system of equations to be solved is then

$$R N_{R,R} + N_R - N_\Theta = 0, \tag{4.26}$$

$$R\, M_{R,R} + M_R - M_\Theta + R\,\Gamma_{,R}\, N_R + (p + \gamma_0\, g)\, \frac{R^2}{2} = 0, \tag{4.27}$$

where

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}, \tag{4.28}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta} + B_\nu\,k_{RR} + B\,k_{\Theta\Theta}, \tag{4.29}$$

$$M_R = -\mathcal{M} - B\,\epsilon^0_{RR} - B_\nu\,\epsilon^0_{\Theta\Theta} - D\,k_{RR} - D_\nu\,k_{\Theta\Theta}, \tag{4.30}$$

$$M_\Theta = -\mathcal{M} - B_\nu\,\epsilon^0_{RR} - B\,\epsilon^0_{\Theta\Theta} - D_\nu\,k_{RR} - D\,k_{\Theta\Theta}, \tag{4.31}$$

and

$$\epsilon^0_{RR} = u_{,R} + \Gamma_{,R} w_{,R}, \qquad k_{RR} = w_{,RR}, \tag{4.32}$$

$$\epsilon^0_{\Theta\Theta} = \frac{u}{R}, \qquad k_{\Theta\Theta} = \frac{w_{,R}}{R}. \tag{4.33}$$

Next, as in §3.2, we transform these equations to dimensionless forms by introducing a dimensionless coordinate ρ, dimensionless displacement components $\widehat{u}$ and $\widehat{w}$, and a dimensionless initially curved reference surface $\widehat{\Gamma}$, defined by

$$\rho \equiv \frac{R}{a}, \qquad \widehat{u} \equiv \frac{u}{a}, \qquad \widehat{w} \equiv \frac{w}{a}, \qquad \widehat{\Gamma} \equiv \frac{\Gamma}{a}, \tag{4.34}$$

together with the following dimensionless constants:

$$\nu_A \equiv \frac{A_\nu}{A}, \qquad \overline{E} \equiv \frac{A}{h}\left(1 - \nu_A^2\right), \qquad \tau \equiv \frac{\mathcal{N}}{\overline{E}\,h}, \qquad \mu \equiv \frac{\mathcal{M}}{\overline{E}\,h\,a}, \qquad q \equiv \frac{(p + \gamma_0 g)\,a}{\overline{E}\,h}, \tag{4.35}$$

and four dimensionless dependent variables x_r, x_θ, y_r, and y_θ defined by

$$x_r \equiv \frac{N_R - \mathcal{N}}{\overline{E}\,h}, \qquad x_\theta \equiv \frac{N_\Theta - \mathcal{N}}{\overline{E}\,h}, \qquad y_r \equiv \frac{M_R + \mathcal{M}}{\overline{E}\,h\,a}, \qquad y_\theta \equiv \frac{M_\Theta + \mathcal{M}}{\overline{E}\,h\,a}. \tag{4.36}$$

We also introduce dimensionless curvatures κ_r and κ_θ defined by

$$\kappa_r \equiv a\,k_{RR} = \widehat{w}_{,\rho\rho} \qquad \kappa_\theta \equiv a\,k_{\Theta\Theta} = \frac{\widehat{w}_{,\rho}}{\rho}, \tag{4.37}$$

and four new dimensionless constants

$$b \equiv \frac{B}{\overline{E}\,h\,a}, \qquad b_\nu \equiv \frac{B_\nu}{\overline{E}\,h\,a}, \qquad d \equiv \frac{D}{\overline{E}\,h\,a^2}, \qquad d_\nu \equiv \frac{D_\nu}{\overline{E}\,h\,a^2}, \tag{4.38}$$

noting that the strain components are already dimensionless, and have the following forms in terms of the dimensionless variables defined above:

$$\epsilon^0_{RR} = \widehat{u}_{,\rho} + \widehat{\Gamma}_{,\rho}\,\widehat{w}_{,\rho}, \qquad \epsilon^0_{\Theta\Theta} = \frac{\widehat{u}}{\rho}. \tag{4.39}$$

The strain components must satisfy a modified form of the compatibility condition (3.40), viz.,

$$\rho\, \epsilon^0_{\Theta\Theta,\rho} = \epsilon^0_{RR} - \epsilon^0_{\Theta\Theta} - \widehat{\Gamma}_{,\rho}\, \widehat{w}_{,\rho}, \tag{4.40}$$

and there is an additional compatibility condition involving the dimensionless curvatures:

$$\rho\, \kappa_{\theta,\rho} = \kappa_r - \kappa_\theta, \tag{4.41}$$

which is easily derived from the definitions (4.37).

Substituting these dimensionless quantities into the equilibrium equations (4.26) and (4.27), we reduce them to

$$x_{r,\rho} + \frac{1}{\rho}(x_r - x_\theta) = 0, \tag{4.42}$$

and

$$y_{r,\rho} + \frac{1}{\rho}(y_r - y_\theta) + \widehat{\Gamma}_{,\rho}(\tau + x_r) + q\,\frac{\rho}{2} = 0, \tag{4.43}$$

respectively. The constitutive relations (4.28)–(4.31) take the forms

$$x_r = \frac{1}{1-\nu_A^2}\left(\epsilon^0_{RR} + \nu_A\, \epsilon^0_{\Theta\Theta}\right) + b\, \kappa_r + b_\nu\, \kappa_\theta, \tag{4.44}$$

$$x_\theta = \frac{1}{1-\nu_A^2}\left(\nu_A\, \epsilon^0_{RR} + \epsilon^0_{\Theta\Theta}\right) + b_\nu\, \kappa_r + b\, \kappa_\theta, \tag{4.45}$$

$$y_r = -b\, \epsilon^0_{RR} - b_\nu\, \epsilon^0_{\Theta\Theta} - d\, \kappa_r - d_\nu\, \kappa_\theta, \tag{4.46}$$

$$y_\theta = -b_\nu\, \epsilon^0_{RR} - b\, \epsilon^0_{\Theta\Theta} - d_\nu\, \kappa_r - d\, \kappa_\theta. \tag{4.47}$$

Equations (4.44) and (4.45) are easily solved for ϵ^0_{RR} and $\epsilon^0_{\Theta\Theta}$ in terms of the x's and κ's:

$$\epsilon^0_{RR} = x_r - \nu_A\, x_\theta - \beta\, \kappa_r - \beta_\nu\, \kappa_\theta, \tag{4.48}$$

$$\epsilon^0_{\Theta\Theta} = x_\theta - \nu_A\, x_r - \beta_\nu\, \kappa_r - \beta\, \kappa_\theta, \tag{4.49}$$

where we have introduced two new constants:

$$\beta \equiv b - \nu_A\, b_\nu, \qquad \beta_\nu \equiv b_\nu - \nu_A\, b. \tag{4.50}$$

We note here that the dimensionless radial displacement component can be expressed in terms of x_r and x_θ, and κ_r and κ_θ (which themselves depend only on derivatives of $\widehat{w}$), by combining (4.39) and (4.49) to obtain

$$\widehat{u}(\rho) = \rho\,(x_\theta - \nu_A\, x_r - \beta_\nu\, \kappa_r - \beta\, \kappa_\theta). \tag{4.51}$$

Taking the derivative with respect to ρ of (4.49) and substituting the result, together with (4.48) and (4.49), in the ϵ-compatibility relation (4.40), we obtain

$$\rho\,(x_{\theta,\rho} - \nu_A\, x_{r,\rho} - \beta_\nu\, \kappa_{r,\rho} - \beta\, \kappa_{\theta,\rho}) = (1 + \nu_A)(x_r - x_\theta) - (\beta - \beta_\nu)(\kappa_r - \kappa_\theta) - \widehat{\Gamma}_{,\rho}\, \widehat{w}_{,\rho}.$$

In this expression, we use (4.41) to replace $(\kappa_r - \kappa_\theta)$, and (4.42) to replace $(x_r - x_\theta)$, which yields the following result for the compatibility condition:

$$\rho\left[x_r + x_\theta - \beta_\nu\,(\kappa_r + \kappa_\theta)\right]_{,\rho} + \widehat{\Gamma}_{,\rho}\, \widehat{w}_{,\rho} = 0. \tag{4.52}$$

Next, we substitute (4.48) and (4.49) in (4.46) and (4.47) to obtain

$$y_r = -\beta\, x_r - \beta_\nu\, x_\theta - \delta\, \kappa_r - \delta_\nu\, \kappa_\theta, \tag{4.53}$$

$$y_\theta = -\beta_\nu\, x_r - \beta\, x_\theta - \delta_\nu\, \kappa_r - \delta\, \kappa_\theta, \tag{4.54}$$

where we introduce another pair of constants:

$$\delta \equiv d - b\beta - b_\nu\beta_\nu, \qquad \delta_\nu \equiv d_\nu - b\beta_\nu - b_\nu\beta. \tag{4.55}$$

The derivative of y_r with respect to ρ yields from (4.53):

$$y_{r,\rho} = -\beta\, x_{r,\rho} - \beta_\nu\, x_{\theta,\rho} - \delta\, \kappa_{r,\rho} - \delta_\nu\, \kappa_{\theta,\rho}. \tag{4.56}$$

From equations (4.53), (4.54), and (4.56) we obtain

$$\begin{aligned} y_{r,\rho} + \frac{1}{\rho}(y_r - y_\theta) &= -\beta\, x_{r,\rho} - \beta_\nu\, x_{\theta,\rho} - \delta\, \kappa_{r,\rho} - \delta_\nu\, \kappa_{\theta,\rho} \\ &\quad - \frac{1}{\rho}\left[(\beta - \beta_\nu)(x_r - x_\theta) + (\delta - \delta_\nu)(\kappa_r - \kappa_\theta)\right]. \end{aligned}$$

In this expression, we again use (4.41) to replace $(\kappa_r - \kappa_\theta)$, and (4.42) to replace $(x_r - x_\theta)$, which simplifies it to

$$y_{r,\rho} + \frac{1}{\rho}(y_r - y_\theta) = -\beta_\nu\, (x_r + x_\theta)_{,\rho} - \delta\, (\kappa_r + \kappa_\theta)_{,\rho}. \tag{4.57}$$

Substitution of (4.57) into (4.43) yields the following form of the axial equilibrium equation:

$$-\beta_\nu\, (x_r + x_\theta)_{,\rho} - \delta\, (\kappa_r + \kappa_\theta)_{,\rho} + \widehat{\Gamma}_{,\rho}\, (\tau + x_r) + q\,\frac{\rho}{2} = 0. \tag{4.58}$$

Equation (4.42) can be used to eliminate x_θ in (4.52) and (4.58), and the problem we are left with is to solve the coupled differential equations (4.52) and (4.58) for $\widehat{w}$ and x_r. The radial displacement $\widehat{u}$ is then determined from (4.51).

Equations (4.52) and (4.58) differ in form from those for a shell of a *single* material *only* by the terms involving the coefficient β_ν. From its definition, given in (4.50), we note that

$$\beta_\nu = b_\nu - \nu_A\, b = \frac{B}{\bar{E}\,h\,a}\,(\nu_B - \nu_A), \tag{4.59}$$

i.e., β_ν is proportional to the difference in extensional and coupling Poisson's ratios. This expression can be expanded, yielding

$$\begin{aligned} \beta_\nu &= \frac{1}{A\bar{E}\,h\,a}\,(A\,B_\nu - A_\nu\,B) \\ &= \frac{1}{A\bar{E}\,h\,a}\,\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} h_i\,h_j\,h_k\,\left[Q_i\,(Q_j\nu_j - Q_k\nu_k) - Q_i\nu_i\,(Q_j - Q_k)\right], \end{aligned}$$

which can be manipulated to

$$\begin{aligned} \beta_\nu = \frac{1}{A\bar{E}\,h\,a}\Big\{ &\frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} h_i\,h_j\,h_k\,Q_i\,Q_k\,(\nu_i - \nu_k) \\ &+ \frac{h_N\,Q_N}{2}\sum_{j=1}^{N-1}\sum_{k=j+1}^{N} h_j\,h_k\,\left[Q_j\,(\nu_j - \nu_N) - Q_k\,(\nu_k - \nu_N)\right]\Big\}. \end{aligned} \tag{4.60}$$

This constant is seen to be a sum of terms, each of which depends upon *differences in the Poisson's ratios* of the various material layers. For a single-layer coating of the membrane (hence $N = 2$ layers total), we have simply

$$\beta_\nu = \frac{h_c\, h_s\, Q_c\, Q_s}{2\, A\, \overline{E}\, a} \, (\nu_c - \nu_s)\,, \tag{4.61}$$

where $i = j = 1 = c$ in the coating, and $k = 2 = N = s$ in the membrane substrate. The total thickness $h = h_c + h_s$ disappears from the expression, as it is a common factor in both numerator and denominator. Thus, if the two materials have the same Poisson's ratio, β_ν vanishes and the governing equations simplify considerably (this simplifying assumption was made, for example, by Wittrick [18] in his work on the stability of bimetallic thermostats).

4.2 General Solution for an Initially Parabolic Coated Membrane Laminate

Here, we consider an initially parabolic coated membrane, defined by equation (1.19), viz.,

$$\Gamma(R) = \frac{\kappa}{2}\left(a^2 - R^2\right), \tag{4.62}$$

The dimensionless form of this equation can be written as

$$\widehat{\Gamma}(\rho) = \frac{\kappa_o}{2}\left(1 - \rho^2\right), \tag{4.63}$$

where κ_o is a dimensionless parameter defined by

$$\kappa_o \equiv a\,\kappa = \frac{a}{2f}. \tag{4.64}$$

From (4.63) we obtain

$$\widehat{\Gamma}_{,\rho} = -\kappa_o\,\rho, \tag{4.65}$$

which can be substituted in the fundamental equations (4.52) and (4.58) to write them as

$$\left[x_r + x_\theta - \beta_\nu\,(\kappa_r + \kappa_\theta)\right]_{,\rho} - \kappa_o\,\widehat{w}_{,\rho} = 0, \tag{4.66}$$

$$-\beta_\nu\,(x_r + x_\theta)_{,\rho} - \delta\,(\kappa_r + \kappa_\theta)_{,\rho} - \kappa_o\,\rho\,(\tau + x_r) + q\,\frac{\rho}{2} = 0. \tag{4.67}$$

This pair of equations can be uncoupled to obtain a single differential equation for $\widehat{w}$ as follows. Substituting for $(x_r + x_\theta)_{,\rho}$ from (4.66) into (4.67) yields

$$-\Delta\,(\kappa_r + \kappa_\theta)_{,\rho} - \beta_\nu\,\kappa_o\,\widehat{w}_{,\rho} - \kappa_o\,\rho\,(\tau + x_r) + q\,\frac{\rho}{2} = 0, \tag{4.68}$$

where we have introduced a new constant Δ defined by

$$\Delta \equiv \delta + \beta_\nu^2 = d - \left(1 - \nu_A^2\right) b^2. \tag{4.69}$$

Solving (4.68) for x_r yields (for $\kappa_o \neq 0$)

$$x_r = -\tau - \frac{1}{\kappa_o}\left[\Delta\,\frac{(\kappa_r + \kappa_\theta)_{,\rho}}{\rho} + \beta_\nu\,\kappa_o\,\frac{\widehat{w}_{,\rho}}{\rho} - \frac{q}{2}\right], \tag{4.70}$$

from which

$$x_{r,\rho} = -\frac{\Delta}{\kappa_o}\left[\frac{(\kappa_r + \kappa_\theta)_{,\rho\rho}}{\rho} - \frac{(\kappa_r + \kappa_\theta)_{,\rho}}{\rho^2}\right] - \beta_\nu\left(\frac{\widehat{w}_{,\rho\rho}}{\rho} - \frac{\widehat{w}_{,\rho}}{\rho^2}\right). \tag{4.71}$$

From equations (4.42), (4.70), and (4.71) we have

$$x_\theta = \rho\, x_{r,\rho} + x_r = -\tau - \frac{\Delta}{\kappa_o}\left(\kappa_r + \kappa_\theta\right)_{,\rho\rho} - \beta_\nu\, \widehat{w}_{,\rho\rho} + \frac{q}{2\kappa_o}, \tag{4.72}$$

hence (4.70) and (4.72) yield

$$x_r + x_\theta = -2\tau + \frac{q}{\kappa_o} - \frac{\Delta}{\kappa_o}\left[\left(\kappa_r + \kappa_\theta\right)_{,\rho\rho} + \frac{\left(\kappa_r + \kappa_\theta\right)_{,\rho}}{\rho}\right] - \beta_\nu\left(\widehat{w}_{,\rho\rho} + \frac{\widehat{w}_{,\rho}}{\rho}\right). \tag{4.73}$$

We next observe that equation (4.66) can be integrated immediately to obtain

$$x_r + x_\theta = \beta_\nu\left(\kappa_r + \kappa_\theta\right) + \kappa_o\, \widehat{w} + C_3, \tag{4.74}$$

where C_3 is an arbitrary integration constant. Substituting from (4.73) into (4.74) yields

$$\frac{\Delta}{\kappa_o}\left[\left(\kappa_r + \kappa_\theta\right)_{,\rho\rho} + \frac{\left(\kappa_r + \kappa_\theta\right)_{,\rho}}{\rho}\right] + \beta_\nu\left(\widehat{w}_{,\rho\rho} + \frac{\widehat{w}_{,\rho}}{\rho} + \kappa_r + \kappa_\theta\right) + \kappa_o\, \widehat{w} + 2\tau - \frac{q}{\kappa_o} + C_3 = 0. \tag{4.75}$$

Recalling the definition of the ρ-dependent part of the Laplacian operator in cylindrical coordinates, viz.,

$$\nabla^2 \widehat{w} \equiv \widehat{w}_{,\rho\rho} + \frac{\widehat{w}_{,\rho}}{\rho}, \tag{4.76}$$

we have

$$\kappa_r + \kappa_\theta = \widehat{w}_{,\rho\rho} + \frac{\widehat{w}_{,\rho}}{\rho} = \nabla^2 \widehat{w}, \tag{4.77}$$

and also

$$\nabla^2\left(\nabla^2 \widehat{w}\right) \equiv \nabla^4 \widehat{w} = \left(\nabla^2 \widehat{w}\right)_{,\rho\rho} + \frac{\left(\nabla^2 \widehat{w}\right)_{,\rho}}{\rho}. \tag{4.78}$$

These observations allow us to write equation (4.75) more compactly as

$$\nabla^4 \widehat{w} + 2\,\beta_\nu \frac{\kappa_o}{\Delta} \nabla^2 \widehat{w} + \frac{\kappa_o^2}{\Delta}\, \widehat{w} = \frac{1}{\Delta}\left(q - 2\tau\kappa_o - C_3\,\kappa_o\right). \tag{4.79}$$

The complete solution of the linear differential equation (4.79) has the form $\widehat{w}(R) = \widehat{w}_h(R) + \widehat{w}_p(R)$, where $\widehat{w}_h(R)$ is the general solution of the *homogeneous* equation

$$\nabla^4 \widehat{w}_h + 2\,\beta_\nu \frac{\kappa_o}{\Delta} \nabla^2 \widehat{w}_h + \frac{\kappa_o^2}{\Delta}\, \widehat{w}_h = 0, \tag{4.80}$$

and $\widehat{w}_p(R)$ is a *particular* solution of (4.79). A suitable particular solution in this case is the *constant* function

$$\widehat{w}_p(R) = \frac{1}{\kappa_o^2}\left(q - 2\tau\kappa_o - C_3\,\kappa_o\right), \tag{4.81}$$

which involves the as yet unknown integration constant C_3. The homogeneous fourth-order linear differential equation (4.80) has at most four linearly independent solutions. We note that if $\psi(\rho)$ is an eigenfunction of the ρ-dependent Laplacian operator defined by (4.76), corresponding to an eigenvalue $-m^2$, i.e., if

$$\nabla^2 \psi = -m^2 \psi, \tag{4.82}$$

then $\nabla^4 \psi = m^4 \psi$, hence ψ is *also* a solution of (4.80) *if* m satisfies the fourth-degree polynomial equation

$$m^4 + 2\,k_2^2\, m^2 + k_1^4 = 0, \tag{4.83}$$

where we have introduced two constants k_2 and k_1 defined by

$$k_2^2 \equiv -\beta_\nu \frac{\kappa_o}{\Delta}, \qquad k_1^4 \equiv \frac{\kappa_o^2}{\Delta}, \tag{4.84}$$

and remark that β_ν can be either positive or negative, while Δ is always positive. The four solutions of (4.83) are easily found to be

$$m = \pm\sqrt{-k_2^2 \pm \sqrt{k_2^4 - k_1^4}}\;. \tag{4.85}$$

The forms of these solutions to be used depends on the relation between k_2^2 and k_1^2. We find for typical parameter values of the material constants and the geometry that $k_1^2 > k_2^2$, so that (4.85) may be written as

$$m = \pm\sqrt{-k_2^2 \pm i\sqrt{(k_1^2 + k_2^2)(k_1^2 - k_2^2)}}\;. \tag{4.86}$$

This can be put into a more convenient form by introducing two new (real) constants defined by

$$k \equiv \sqrt{\frac{1}{2}(k_1^2 + k_2^2)}\;, \qquad \epsilon \equiv \sqrt{\frac{1}{2}(k_1^2 - k_2^2)}\;, \tag{4.87}$$

which can be inverted to yield k_2^2 and k_1^2 in terms of ϵ and k:

$$k_2^2 = k^2 - \epsilon^2, \qquad k_1^2 = k^2 + \epsilon^2. \tag{4.88}$$

The complex eigenvalues (4.86) take the following simple forms in terms of ϵ and k:

$$m = \pm(\epsilon \pm ik)\,. \tag{4.89}$$

The four distinct eigenvalues m_j, $j = 1, 2, 3, 4$, are thus given by

$$\begin{aligned} m_1 &= \epsilon + ik, \\ m_2 &= \epsilon - ik \equiv m_1^*, \\ m_3 &= -(\epsilon + ik) \equiv -m_1, \\ m_4 &= -(\epsilon - ik) \equiv m_3^* \equiv -m_2 \equiv -m_1^*, \end{aligned} \tag{4.90}$$

where an asterisk denotes *complex conjugation.* These correspond to eigenfunctions $\psi_j(\rho)$ satisfying

$$\nabla^2\psi_j = -m_j^2\psi_j, \qquad j = 1, 2, 3, 4. \tag{4.91}$$

Substituting for the Laplacian operator, we obtain the following second-order differential equation for each ψ_j:

$$\psi_j'' + \frac{\psi_j'}{\rho} + m_j^2\psi_j = 0. \tag{4.92}$$

Introducing new independent (complex) variables $\xi_j \equiv m_j\,\rho$, the jth second-order equation can be written as

$$\varphi_j'' + \frac{\varphi_j'}{\xi_j} + \varphi_j = 0, \tag{4.93}$$

where a prime always denotes the derivative of a function with respect to its argument, in this case ξ_j, and $\varphi_j(\xi_j) \equiv \psi_j(m_j\,\rho)$. Equation (4.93) is Bessel's equation of zero order, whose solution *regular at the origin* is the zero-order Bessel function $J_0(\xi_j)$. The general solution of our homogeneous equation (4.80) is then a linear combination of the four eigenfunctions $J_0(\xi_j)$:

$$\begin{aligned} \widehat{w}_h(\rho) &= \widetilde{C}_1 J_0(m_1\rho) + \widetilde{C}_2 J_0(m_2\rho) + \widetilde{C}_3 J_0(m_3\rho) + \widetilde{C}_4 J_0(m_4\rho), \\ &= \widetilde{C}_1 J_0(m_1\rho) + \widetilde{C}_2 J_0(m_1^*\rho) + \widetilde{C}_3 J_0(-m_1\rho) + \widetilde{C}_4 J_0(-m_1^*\rho), \end{aligned} \tag{4.94}$$

where the arbitrary constants $\widetilde{C}_j$ are, in general, complex. Using the fact that J_0 is even, i.e., $J_0(-z) = J_0(z)$, this homogeneous solution reduces to

$$\widehat{w}_h(\rho) = \overline{C}_1 J_0(m_1\rho) + \overline{C}_2 J_0(m_1^*\rho), \tag{4.95}$$

where $\overline{C}_1 \equiv \widetilde{C}_1 + \widetilde{C}_3$ and $\overline{C}_2 \equiv \widetilde{C}_2 + \widetilde{C}_4$ are new arbitrary complex constants. From its definition in (4.90), we can write the complex number m_1 in polar form as

$$m_1 = \epsilon + ik = \sqrt{\epsilon^2 + k^2}\, e^{i\theta_1} = k_1\, e^{i\theta_1}, \qquad \theta_1 = \tan^{-1}\left(\frac{k}{\epsilon}\right), \tag{4.96}$$

where we also used the second equation of (4.88). Equation (4.95) is thus equivalent to

$$\widehat{w}_h(\rho) = \overline{C}_1 J_0(e^{i\theta_1} k_1\rho) + \overline{C}_2 J_0(e^{-i\theta_1} k_1\rho). \tag{4.97}$$

Now, the ratio k/ϵ defining the angle θ_1 can be rewritten using (4.87) and (4.84) as

$$\frac{k}{\epsilon} = \sqrt{\frac{1 + \beta_\nu\, \Delta^{-1/2}}{1 - \beta_\nu\, \Delta^{-1/2}}}. \tag{4.98}$$

We find that for the material and geometric parameters of interest here, it is in fact true that $|\beta_\nu\, \Delta^{-1/2}|$ is always much less than 1 (in a typical case, this product is on the order of 1×10^{-8}). We thus approximate

$$\frac{k}{\epsilon} \approx 1, \quad \Rightarrow \quad \theta_1 \approx \frac{\pi}{4}, \tag{4.99}$$

which brings (4.97) to the form

$$\widehat{w}_h(\rho) = \overline{C}_1 J_0(e^{i\pi/4} k_1\rho) + \overline{C}_2 J_0(e^{-i\pi/4} k_1\rho). \tag{4.100}$$

This can be written in terms of Kelvin's *ber* and *bei* functions (see [19, pp. 379-385], for details of the properties of the Kelvin functions) as

$$\widehat{w}_h(\rho) = \overline{C}_1 \left[\mathrm{ber}(k_1\rho) + i\,\mathrm{bei}(k_1\rho)\right] + \overline{C}_2 \left[\mathrm{ber}(k_1\rho) - i\,\mathrm{bei}(k_1\rho)\right], \tag{4.101}$$

Since Kelvin's functions are real-valued, and $\widehat{w}_h(\rho)$ must be real, it follows that the complex constants must in fact be complex conjugates: $\overline{C}_2 = \overline{C}_1^*$. Thus, the general solution for $\widehat{w}(\rho)$ takes the form

$$\widehat{w}(\rho) = \widehat{w}_h(\rho) + \widehat{w}_p(\rho) = C_1 \mathrm{ber}(k_1\rho) + C_2 \mathrm{bei}(k_1\rho) + \frac{1}{\kappa_o^2}\left(q - 2\tau\,\kappa_o - C_3\,\kappa_o\right), \tag{4.102}$$

where $C_1 \equiv \overline{C}_1 + \overline{C}_1^*$ and $C_2 \equiv i(\overline{C}_1 - \overline{C}_1^*)$ are now arbitrary *real* constants to be determined from the boundary conditions.

The dimensionless radial displacement $\widehat{u}$ defined by (4.51) can be expressed in terms of $\widehat{w}$ and its derivatives as follows. First, we use (4.70) and (4.72) to replace x_r and x_θ in (4.51), yielding

$$\widehat{u}(\rho) = \rho\left\{\left(\frac{q}{2\kappa_o} - \tau\right)(1 - \nu_A) - \frac{\Delta}{\kappa_o}\left[\left(\nabla^2\widehat{w}\right)_{,\rho\rho} - \nu_A\,\frac{\left(\nabla^2\widehat{w}\right)_{,\rho}}{\rho}\right] - \beta_\nu\left(2\,\widehat{w}_{,\rho\rho} - \nu_A\,\frac{\widehat{w}_{,\rho}}{\rho}\right) - \beta\,\frac{\widehat{w}_{,\rho}}{\rho}\right\}.$$

From (4.79) and (4.78) we obtain the following expression for $\left(\nabla^2\widehat{w}\right)_{,\rho\rho}$:

$$\nabla^2\widehat{w}_{,\rho\rho} = -\frac{\left(\nabla^2\widehat{w}\right)_{,\rho}}{\rho} - 2\,\beta_\nu\,\frac{\kappa_o}{\Delta}\,\nabla^2\widehat{w} - \frac{\kappa_o^2}{\Delta}\,\widehat{w} - 2\,\tau\,\frac{\kappa_o}{\Delta} + \frac{q}{\Delta} - C_3\,\frac{\kappa_o}{\Delta},$$

which is substituted for $\left(\nabla^2 \widehat{w}\right)_{,\rho\rho}$ in the previous equation to obtain the desired result, viz.,

$$\widehat{u}(\rho) = \left[\left(\tau - \frac{q}{2\kappa_o}\right)(1 + \nu_A) + C_3\right]\rho + (1 + \nu_A)\,\frac{\Delta}{\kappa_o}\,\left(\nabla^2 \widehat{w}\right)_{,\rho} + \left[\beta_\nu\,(2 + \nu_A) - \beta\right]\widehat{w}_{,\rho} + \kappa_o\,\rho\,\widehat{w}\,. \quad (4.103)$$

To get the derivatives of $\widehat{w}(\rho)$ appearing in (4.103), in terms of Kelvin functions, we note that

$$\widehat{w}_{,\rho}(\rho) = k_1\left[C_1 \mathrm{ber}'(k_1\rho) + C_2 \mathrm{bei}'(k_1\rho)\right], \quad (4.104)$$

$$\widehat{w}_{,\rho\rho}(\rho) = k_1^2\left[C_1 \mathrm{ber}''(k_1\rho) + C_2 \mathrm{bei}''(k_1\rho)\right], \quad (4.105)$$

where the prime on the Kelvin functions here denotes a derivative with respect to the function argument $x \equiv k_1\rho$. From [19], Equations (9.9.16),

$$\mathrm{ber}'(x) = \frac{1}{\sqrt{2}}\left[\ \mathrm{ber}_1(x) + \mathrm{bei}_1(x)\right], \quad (4.106)$$

$$\mathrm{bei}'(x) = \frac{1}{\sqrt{2}}\left[-\mathrm{ber}_1(x) + \mathrm{bei}_1(x)\right], \quad (4.107)$$

hence

$$\mathrm{ber}''(x) = \frac{1}{\sqrt{2}}\left[\ \mathrm{ber}_1'(x) + \mathrm{bei}_1'(x)\right], \quad (4.108)$$

$$\mathrm{bei}''(x) = \frac{1}{\sqrt{2}}\left[-\mathrm{ber}_1'(x) + \mathrm{bei}_1'(x)\right], \quad (4.109)$$

For the derivatives of the Kelvin functions of order 1, we have from Equations (9.9.14) of [19] the following identities:

$$\mathrm{ber}_1'(x) = \frac{\sqrt{2}}{4}\left[\mathrm{ber}_2(x) + \mathrm{bei}_2(x) - \mathrm{ber}(x) - \mathrm{bei}(x)\right], \quad (4.110)$$

$$\mathrm{bei}_1'(x) = \frac{\sqrt{2}}{4}\left[\mathrm{bei}_2(x) - \mathrm{ber}_2(x) - \mathrm{bei}(x) + \mathrm{ber}(x)\right], \quad (4.111)$$

and

$$\mathrm{ber}_2(x) = -\frac{\sqrt{2}}{x}\left[\mathrm{ber}_1(x) - \mathrm{bei}_1(x)\right] - \mathrm{ber}(x), \quad (4.112)$$

$$\mathrm{bei}_2(x) = -\frac{\sqrt{2}}{x}\left[\mathrm{bei}_1(x) + \mathrm{ber}_1(x)\right] - \mathrm{bei}(x). \quad (4.113)$$

Applying the last four identities to (4.108) and (4.109) yields

$$\mathrm{ber}''(x) = -\frac{1}{x}\mathrm{ber}'(x) - \mathrm{bei}(x), \quad (4.114)$$

$$\mathrm{bei}''(x) = -\frac{1}{x}\mathrm{bei}'(x) + \mathrm{ber}(x). \quad (4.115)$$

Substituting these results in (4.105), we obtain

$$\widehat{w}_{,\rho\rho}(\rho) = -\frac{k_1}{\rho}\left[C_1 \mathrm{ber}'(k_1\rho) + C_2 \mathrm{bei}'(k_1\rho)\right] + k_1^2\left[-C_1\,\mathrm{bei}(k_1\rho) + C_2\,\mathrm{ber}(k_1\rho)\right] \quad (4.116)$$

$$= -\frac{\widehat{w}_{,\rho}}{\rho} + k_1^2\left[-C_1\,\mathrm{bei}(k_1\rho) + C_2\,\mathrm{ber}(k_1\rho)\right] \quad (4.117)$$

where we used (4.104) in (4.116) to get (4.117), and recall that $x = k_1\rho$. From (4.117) follows a useful form of the ρ-dependent Laplacian operator acting on $\widehat{w}$:

$$\nabla^2\widehat{w} = \widehat{w}_{,\rho\rho} + \frac{\widehat{w}_{,\rho}}{\rho} = k_1^2\left[-C_1\mathrm{bei}(k_1\rho) + C_2\mathrm{ber}(k_1\rho)\right]. \tag{4.118}$$

The derivative of the last expression, which is required in (4.103), can be written as

$$\left(\nabla^2\widehat{w}\right)_{,\rho} = k_1^3\left[-C_1\,\mathrm{bei}'(k_1\rho) + C_2\,\mathrm{ber}'(k_1\rho)\right]. \tag{4.119}$$

Substituting (4.118), (4.119), and (4.102) in (4.103) yields the general solution for $\widehat{u}(\rho)$ in terms of Kelvin functions (the terms containing C_3 conveniently add to zero):

$$\begin{aligned}\widehat{u}(\rho) = {} & u_1\,k_1^3\left[-C_1\mathrm{bei}'(k_1\rho) + C_2\mathrm{ber}'(k_1\rho)\right] + u_2\,k_1\left[C_1\mathrm{ber}'(k_1\rho) + C_2\mathrm{bei}'(k_1\rho)\right] \\ & + \kappa_o\,\rho\left[C_1\mathrm{ber}(k_1\rho) + C_2\mathrm{bei}(k_1\rho)\right] - (1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right)\rho,\end{aligned} \tag{4.120}$$

where we have introduced new constants u_1 and u_2 defined by

$$u_1 \equiv (1 + \nu_A)\,\frac{\Delta}{\kappa_o} \tag{4.121}$$

$$u_2 \equiv \beta_\nu\,(2 + \nu_A) - \beta. \tag{4.122}$$

Before considering specific boundary conditions, we note that for any such conditions that include a specification $\widehat{w}(\rho_0) = 0$ at some point $\rho = \rho_0$ (where ρ_0 is typically either 0 or 1), we will have from (4.102) a condition of the form:

$$0 = C_1\mathrm{ber}(k_1\rho_0) + C_2\mathrm{bei}(k_1\rho_0) + \frac{1}{\kappa_o^2}\left(q - 2\,\tau\,\kappa_o - C_3\,\kappa_o\right), \tag{4.123}$$

which can be used to replace the constant particular solution in (4.102). The solution for $\widehat{w}(\rho)$ is thus given, after applying a boundary condition of this type, by

$$\widehat{w}(\rho) = C_1\left[\mathrm{ber}(k_1\rho) - \mathrm{ber}(k_1\rho_0)\right] + C_2\left[\mathrm{bei}(k_1\rho) - \mathrm{bei}(k_1\rho_0)\right]. \tag{4.124}$$

4.2.1 Computing the Kelvin Functions

For arguments of the Kelvin functions satisfying $k_1\rho > 8$ we use the asymptotic forms of the Kelvin functions and their first derivatives, derivable from material given in Abramowitz and Stegun [19] (in particular, their Equations (9.10.1) and (9.10.2) on p. 381, together with Equations (9.9.16) on p. 380), viz.,

$$\mathrm{ber}(k_1\rho) \sim F(k_1\rho)\,\cos\left(k_1\rho/\sqrt{2} - \pi/8\right), \tag{4.125}$$

$$\mathrm{bei}(k_1\rho) \sim F(k_1\rho)\,\sin\left(k_1\rho/\sqrt{2} - \pi/8\right), \tag{4.126}$$

$$\mathrm{ber}'(k_1\rho) \sim F(k_1\rho)\,\cos\left(k_1\rho/\sqrt{2} + \pi/8\right), \tag{4.127}$$

$$\mathrm{bei}'(k_1\rho) \sim F(k_1\rho)\,\sin\left(k_1\rho/\sqrt{2} + \pi/8\right), \tag{4.128}$$

noting that each approximation contains a common multiplicative factor

$$F(k_1\rho) \equiv \frac{\exp\left(k_1\rho/\sqrt{2}\right)}{\sqrt{2\pi k_1\,\rho}}. \tag{4.129}$$

The sometimes large exponential factor $F(k_1\rho)$ may be alternately removed and inserted in constants (defined later) associated with various boundary value problems, in such a way that it will 'divide out', to avoid overflow problems in the computation.

For $-8 \le k_1\rho \le 8$, we use the polynomial approximations given in [19, §9.11, p. 384], i.e.,

$$\mathrm{ber}(k_1\rho) = 1.0 - 64.0x^4 + 113.77778x^8 - 32.36346x^{12} + 2.64191x^{16} - 0.08350x^{20} + 0.00123x^{24} - 0.00001x^{28}, \quad (4.130)$$

$$\mathrm{bei}(k_1\rho) = 16.0x^2 - 113.77778x^6 + 72.81778x^{10} - 10.56766x^{14} + 0.52186x^{18} - 0.01104x^{22} + 0.00011x^{26}, \quad (4.131)$$

$$\mathrm{ber}'(k_1\rho) = 8x\,(-4.0x^2 + 14.22222x^6 - 6.06815x^{10} + 0.66048x^{14} - 0.02609x^{18} + 0.00046x^{22}), \quad (4.132)$$

$$\mathrm{bei}'(k_1\rho) = 8x\,(0.5 - 10.66667x^4 + 11.37778x^8 - 2.31167x^{12} + 1.14677x^{16} - 0.00379x^{20} + 0.00005x^{24}), \quad (4.133)$$

where $x \equiv k_1\rho/8$ on the right-hand sides of these formulas.

4.2.2 Free Edge, Simply-Supported at the Center

The first boundary value problem we consider is that of an initially parabolic coated membrane laminate with a free edge, requiring that $N_R(a) = M_R(a) = 0$, and simply-supported at its center, i.e., $w(0) = 0$. In terms of our dimensionless quantities these translate to $x_r(1) = -\tau$, $y_r(1) = \mu$, and $\widehat{w}(0) = 0$, respectively.

The boundary condition $\widehat{w}(0) = 0$ yields from (4.124) (with $\rho_0 = 0$):

$$\widehat{w}(\rho) = C_1\,[\mathrm{ber}(k_1\rho) - 1] + C_2\mathrm{bei}(k_1\rho), \quad (4.134)$$

since $\mathrm{ber}(0) = 1$ and $\mathrm{bei}(0) = 0$. The remaining two boundary conditions, $x_r(1) = -\tau$ and $y_r(1) = \mu$, provide two linear algebraic equations to be solved for the unknown coefficients C_1 and C_2. The construction of these equations involves computing the strains and curvatures defined in (4.39) and (4.37), viz.,

$$\epsilon^0_{RR} = \widehat{u}_{,\rho} - \kappa_0\,\rho\,\widehat{w}_{,\rho}, \qquad \epsilon^0_{\Theta\Theta} = \frac{\widehat{u}}{\rho}, \qquad \kappa_r = \widehat{w}_{,\rho\rho}, \qquad \kappa_\theta = \frac{\widehat{w}_{,\rho}}{\rho}. \quad (4.135)$$

where we replaced $\widehat{\Gamma}_{,\rho} = -\kappa_0\,\rho$ in the first equation of (4.135). All the relevant quantities have been computed, except for $\widehat{u}_{,\rho}$. From (4.120) we obtain

$$\widehat{u}_{,\rho} = u_1\,k_1^4\left[-C_1\mathrm{bei}''(k_1\rho) + C_2\mathrm{ber}''(k_1\rho)\right] + u_2\,k_2^2\left[C_1\mathrm{ber}''(k_1\rho) + C_2\mathrm{bei}''(k_1\rho)\right] + \kappa_0\rho\widehat{w}_{,\rho} + \kappa_0\,[C_1\mathrm{ber}(k_1\rho) + C_2\mathrm{bei}(k_1\rho)] - (1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right),$$

or, using (4.114) and (4.115) to eliminate the second derivatives of the Kelvin functions:

$$\begin{aligned}\widehat{u}_{,\rho} = {} & u_1\frac{k_1^3}{\rho}\,[C_1\mathrm{bei}'(k_1\rho) - C_2\mathrm{ber}'(k_1\rho)] - u_1k_1^4\,[C_1\mathrm{ber}(k_1\rho) + C_2\mathrm{bei}(k_1\rho)] \\ & - u_2\frac{k_1}{\rho}\,[C_1\mathrm{ber}'(k_1\rho) + C_2\mathrm{bei}'(k_1\rho)] - u_2k_1^2\,[C_1\mathrm{bei}(k_1\rho) - C_2\mathrm{ber}(k_1\rho)] \\ & + \kappa_0\rho\widehat{w}_{,\rho} + \kappa_0\,[C_1\mathrm{ber}(k_1\rho) + C_2\mathrm{bei}(k_1\rho)] - (1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right).\end{aligned} \quad (4.136)$$

Using these results, the strains and curvatures are given by

$$\epsilon^0_{RR} = u_1 \frac{k_1^3}{\rho}\left[C_1 \mathrm{bei}'(k_1\rho) - C_2 \mathrm{ber}'(k_1\rho)\right] - u_1 k_1^4 \left[C_1 \mathrm{ber}(k_1\rho) + C_2 \mathrm{bei}(k_1\rho)\right]$$
$$- u_2 \frac{k_1}{\rho}\left[C_1 \mathrm{ber}'(k_1\rho) + C_2 \mathrm{bei}'(k_1\rho)\right] - u_2 k_1^2 \left[C_1 \mathrm{bei}(k_1\rho) - C_2 \mathrm{ber}(k_1\rho)\right]$$
$$+ \kappa_0 \left[C_1 \mathrm{ber}(k_1\rho) + C_2 \mathrm{bei}(k_1\rho)\right] - (1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right), \quad (4.137)$$

$$\epsilon^0_{\Theta\Theta} = u_1 \frac{k_1^3}{\rho}\left[-C_1 \mathrm{bei}'(k_1\rho) + C_2 \mathrm{ber}'(k_1\rho)\right] + u_2 \frac{k_1}{\rho}\left[C_1 \mathrm{ber}'(k_1\rho) + C_2 \mathrm{bei}'(k_1\rho)\right]$$
$$+ \kappa_0 \left[C_1 \mathrm{ber}(k_1\rho) + C_2 \mathrm{bei}(k_1\rho)\right] - (1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right), \quad (4.138)$$

$$\kappa_r = -\frac{k_1}{\rho}\left[C_1 \mathrm{ber}'(k_1\rho) + C_2 \mathrm{bei}'(k_1\rho)\right] + k_1^2\left[-C_1 \mathrm{bei}(k_1\rho) + C_2 \mathrm{ber}(k_1\rho)\right], \quad (4.139)$$
$$\kappa_\theta = \frac{k_1}{\rho}\left[C_1 \mathrm{ber}'(k_1\rho) + C_2 \mathrm{bei}'(k_1\rho)\right]. \quad (4.140)$$

These expressions must be evaluated at $\rho = 1$, and then substituted in the constitutive relations (4.44) and (4.46) with $x_r = -\tau$ and $y_r = \mu$, i.e.,

$$\epsilon^0_{RR} + \nu_A \epsilon^0_{\Theta\Theta} + \left(1 - \nu_A^2\right)(b\kappa_r + b_\nu \kappa_\theta) + \left(1 - \nu_A^2\right)\tau = 0. \quad (4.141)$$

$$b\epsilon^0_{RR} + b_\nu \epsilon^0_{\Theta\Theta} + d\kappa_r + d_\nu \kappa_\theta + \mu = 0, \quad (4.142)$$

The resulting system of two equations for C_1 and C_2 can be written as

$$s_{11}C_1 + s_{12}C_2 + \left(1 - \nu_A^2\right)\frac{q}{2\kappa_o} = 0, \quad (4.143)$$
$$s_{21}C_1 + s_{22}C_2 + (\beta + \beta_\nu)\frac{q}{2\kappa_o} + \mu - (\beta + \beta_\nu)\tau = 0, \quad (4.144)$$

whose solutions are easily found to be

$$C_1 = -\frac{1}{|s|}\left\{\left[\left(1 - \nu_A^2\right)s_{22} - (\beta + \beta_\nu)s_{12}\right]\frac{q}{2\kappa_o} - s_{12}\left[\mu - (\beta + \beta_\nu)\tau\right]\right\}, \quad (4.145)$$
$$C_2 = \frac{1}{|s|}\left\{\left[\left(1 - \nu_A^2\right)s_{21} - (\beta + \beta_\nu)s_{11}\right]\frac{q}{2\kappa_o} - s_{11}\left[\mu - (\beta + \beta_\nu)\tau\right]\right\}, \quad (4.146)$$

where $|s| = s_{11}s_{22} - s_{12}s_{21}$ is the determinant of the 2×2 coefficient matrix. The matrix elements are rather complicated expressions involving the Kelvin functions and material/geometrical parameters. They can be reduced to the following forms:

$$s_{11} = \left[\kappa_0(1 + \nu_A) - k_1^4 u_1\right]\mathrm{ber}(k_1) - k_1^2\left[u_2 + b\left(1 - \nu_A^2\right)\right]\mathrm{bei}(k_1)$$
$$+ k_1^3 u_1 (1 - \nu_A)\mathrm{bei}'(k_1) + k_1(1 - \nu_A)(\beta_\nu - \beta - u_2)\mathrm{ber}'(k_1), \quad (4.147)$$

$$s_{12} = \left[\kappa_0(1 + \nu_A) - k_1^4 u_1\right]\mathrm{bei}(k_1) + k_1^2\left[u_2 + b\left(1 - \nu_A^2\right)\right]\mathrm{ber}(k_1)$$
$$- k_1^3 u_1 (1 - \nu_A)\mathrm{ber}'(k_1) + k_1(1 - \nu_A)(\beta_\nu - \beta - u_2)\mathrm{bei}'(k_1), \quad (4.148)$$

$$s_{21} = \left[\kappa_0\,(b+b_\nu) - b\,k_1^4 u_1\right] \mathrm{ber}(k_1) \;-\; k_1^2(d+bu_2)\,\mathrm{bei}(k_1) \\ +\; k_1^3 u_1\,(b-b_\nu)\,\mathrm{bei}'(k_1) \;-\; k_1\left[d-d_\nu+(b-b_\nu)\,u_2\right]\mathrm{ber}'(k_1), \quad (4.149)$$

$$s_{22} = \left[\kappa_0\,(b+b_\nu) - b\,k_1^4 u_1\right] \mathrm{bei}(k_1) \;+\; k_1^2(d+bu_2)\,\mathrm{ber}(k_1) \\ -\; k_1^3 u_1\,(b-b_\nu)\,\mathrm{ber}'(k_1) \;-\; k_1\left[d-d_\nu+(b-b_\nu)\,u_2\right]\mathrm{bei}'(k_1). \quad (4.150)$$

In the range $k_1\rho > 8$ the asymptotic approximations (4.125)–(4.128) of the Kelvin functions contain a common exponential factor $F(k_1\rho)$, having the value $F(k_1)$ at the edge $\rho = 1$. Here, we see that in the same range $F(k_1)$ is also a common factor of each matrix element s_{ij}, hence its *square* is a factor of the determinant $|s|$. It then follows that the integration constants C_1 and C_2 are proportional to the *reciprocal* of $F(k_1)$. Since both $\widehat{u}$ and $\widehat{w}$ contain products of C_1 and C_2 with the Kelvin functions, we can factor from each occurrence of these functions a term of the form $F(k_1\rho)$, so that a common *ratio* $F(k_1\rho)/F(k_1) = \exp\left[(k_1/\sqrt{2})(\rho-1)\right]/\sqrt{\rho}$ can be factored from these terms. Thus, for values of $k_1\rho > 8$, we replace the Kelvin functions by the trigonometric parts of their asymptotic expansions, and factor out this common ratio wherever possible to avoid computational problems that may occur for large values of the exponential function.

It is rather impractical to consider a pressure difference between the faces of a coated membrane laminate satisfying *free edge* boundary conditions. We thus set the pressure difference $p = 0$, hence the dimensionless constant q depends only on the gravitational field. Ignoring the effects of gravity, we set $q = 0$. The solutions (4.145) and (4.146) for C_1 and C_2 are then observed to contain the *common* factor

$$\mu \;-\; (\beta+\beta_\nu)\;\tau, \qquad (4.151)$$

and this is the sole dependence of C_1 and C_2 on the intrinsic stress loads S_i (occurring in μ and τ). We can thus write the solutions in (4.134) and (4.120) for $\widehat{w}(\rho)$ and $\widehat{u}(\rho)$ (with $q = 0$) as

$$\widehat{w}(\rho) \;=\; [\mu \;-\; (\beta+\beta_\nu)\;\tau]\;\widetilde{w}(\rho), \qquad (4.152)$$

$$\widehat{u}(\rho) \;=\; [\mu \;-\; (\beta+\beta_\nu)\;\tau]\;\widetilde{u}(\rho) \;-\; \tau\;(1\;-\;\nu_A)\;\rho, \qquad (4.153)$$

where $\widetilde{w}(\rho)$ and $\widetilde{u}(\rho)$ are functions that do not depend on the intrinsic stress loads. From (4.152) we see that the axial displacement will vanish if we can choose the intrinsic stress loads and geometrical/material parameters to satisfy $\mu \;-\; (\beta+\beta_\nu)\;\tau \;=\; 0$. In order for the radial displacement to vanish as well, we must also choose them such that $\tau = 0$. This, in turn, requires $\mu = 0$ for the first condition to be satisfied. Thus, for free edge boundary conditions, and ignoring the effects of gravity, the necessary and sufficient conditions for there to be no displacement from the initial paraboloidal shape upon removal from the mold are simply

$$\tau \;=\; 0 \quad \text{and} \quad \mu \;=\; 0 \quad \Rightarrow \quad \mathcal{N} \;=\; 0 \quad \text{and} \quad \mathcal{M} \;=\; 0, \qquad (4.154)$$

that is, the net intrinsic stress resultant and net intrinsic stress couple must both be zero. Note that for a single coating on the membrane, the condition $\mu \;-\; (\beta+\beta_\nu)\;\tau \;=\; 0$, which is sufficient to insure zero axial displacement, reduces to

$$S_c - \mathcal{B}\,S_s \;=\; 0, \quad \text{where} \quad \mathcal{B} \equiv \frac{Q_c\,(1+\nu_c)}{Q_s\,(1+\nu_s)} \;=\; \left(\frac{E_c}{1-\nu_c}\right) \Big/ \left(\frac{E_s}{1-\nu_s}\right), \qquad (4.155)$$

and we made use of the first definition in (1.8). If the residual stresses are thermally induced, so that

$$S_i \;=\; -\left(\frac{E_i}{1-\nu_i}\right)\,\alpha_i\,\Delta T, \qquad (4.156)$$

where α_i is the CTE of either the coating ($i = 1 = c$) or membrane substrate ($i = 2 = s$), then

$$S_c - \mathcal{B}S_s \;=\; -\left(\frac{E_c}{1-\nu_c}\right)\,(\alpha_c \;-\; \alpha_s)\;\Delta T, \qquad (4.157)$$

which vanishes only if the membrane and coating have the same CTE.

Figures 6 and 7, which follow, compare our solutions for the axial and radial displacements with geometrically linear finite element (FE) solutions of the same problem. The model considered here is an initially parabolic membrane with a single coating. It has a radius of a = 10 cm = 0.1 m, and an f-number of 2, i.e., $F^{\#} = 2$, corresponding to $\kappa = 1.25$/m, see equation (1.18). The coating has thickness $h_1 = h_c = 1\mu$m, and the membrane thickness is $h_2 = h_s = 20\mu$m. The coating modulus is taken to be $E_c = 44.0$ GPa, and the membrane modulus $E_s = 2.2$ GPa. We assume the Poisson's ratios of the coating and membrane to be the same, viz., $\nu_c = \nu_s = 0.4$. The assumed intrinsic stresses in this example are small: $S_c = -5$ KPa and $S_s = 20$ Pa.

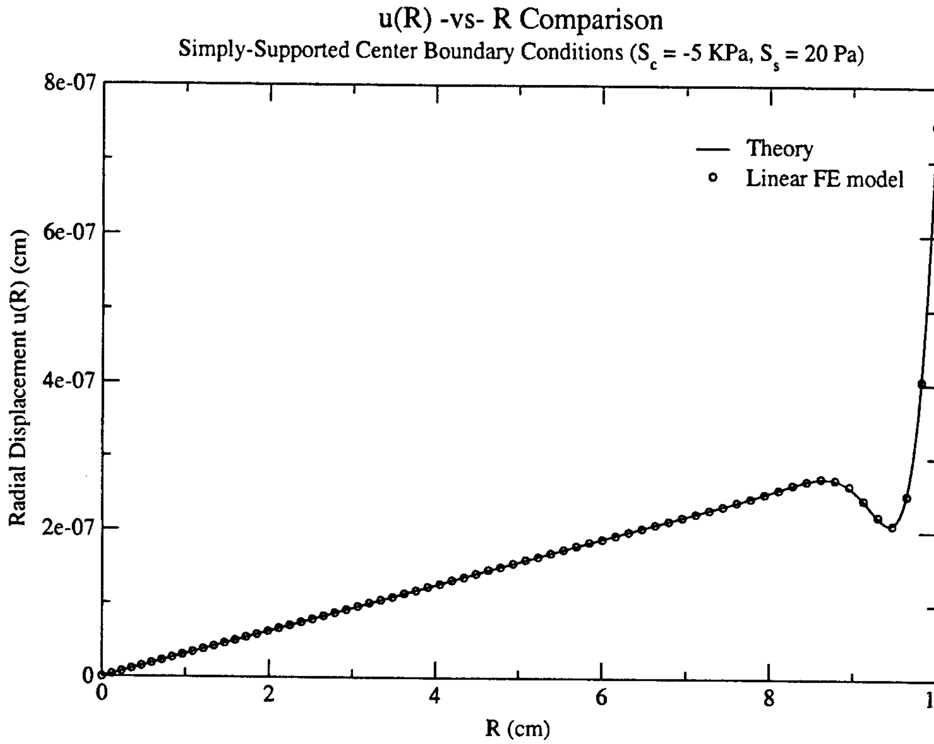

Figure 6: Comparison of theory to geometrically linear FE u-displacement results (free edge/simply-supported at the center).

4.2.3 Pinned, or Hinged, Edge

The boundary conditions for a coated membrane with pinned (or hinged) edge are $w(a) = 0$, $u(a) = 0$, and $M_R(a) = 0$. In terms of the dimensionless coordinate ρ, we have $\widehat{w}(1) = 0$, so that $\rho_0 = 1$ in equation (4.124), $\widehat{u}(1) = 0$ and $y_r(1) = \mu$. The first condition yields from (4.124):

$$\widehat{w}(\rho) = C_1 \left[\mathrm{ber}(k_1\rho) - \mathrm{ber}(k_1)\right] + C_2 \left[\mathrm{bei}(k_1\rho) - \mathrm{bei}(k_1)\right]. \tag{4.158}$$

The solution for $\widehat{u}(\rho)$ is given by (4.120), except that the coefficients C_1 and C_2 are not necessarily the same as those in (4.120), as we shall soon see.

The remaining two boundary conditions, $\widehat{u}(1) = 0$ and $y_r(1) = \mu$, again provide a system of two equations for the constants C_1 and C_2. We note that since $\widehat{u}(\rho)$ and $\widehat{w}_{,\rho}(\rho)$ have precisely the same forms as in the

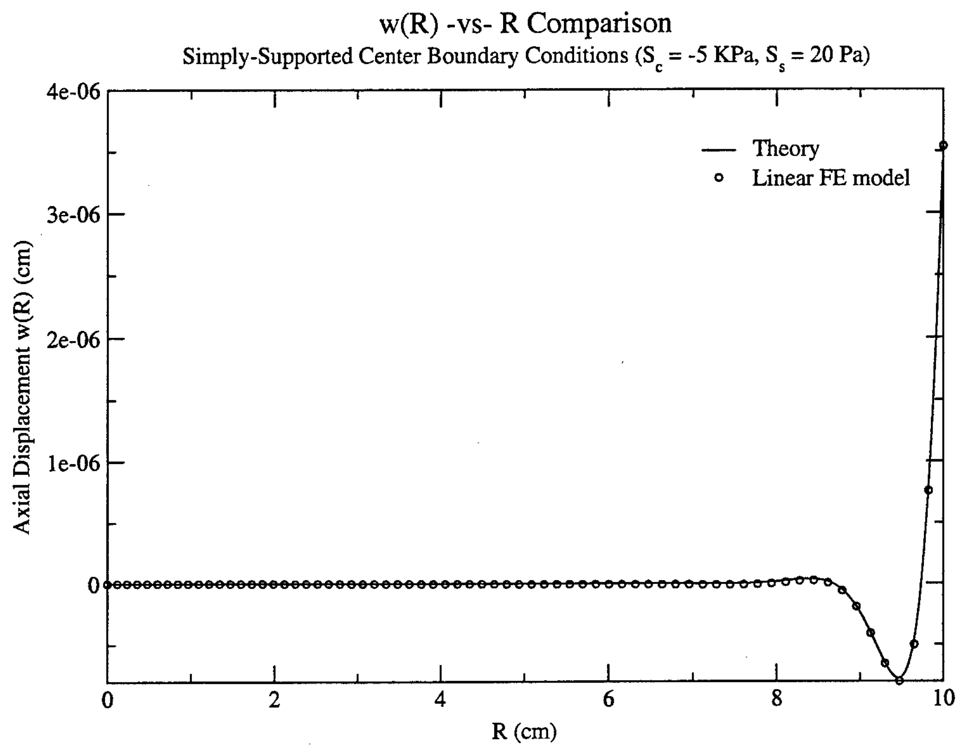

Figure 7: Comparison of theory to geometrically linear FE w-displacement results (free edge/simply-supported at the center).

simply-supported center problem, $y_r(1) = \mu$ duplicates equation (4.142):

$$b\,\epsilon^0_{RR} + b_\nu\,\epsilon^0_{\Theta\Theta} + d\,\kappa_r + d_\nu\,\kappa_\theta + \mu = 0. \tag{4.159}$$

The equation $\widehat{u}(1) = 0$, however, is new and this, together with (4.159), leads to the following system of equations:

$$p_{11}C_1 + p_{12}C_2 - (1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right) = 0, \tag{4.160}$$

$$p_{21}C_1 + p_{22}C_2 + \mu - (\beta + \beta_\nu)\left(\tau - \frac{q}{2\kappa_o}\right) = 0, \tag{4.161}$$

where $p_{21} = s_{21}$ and $p_{22} = s_{22}$, given in equations (4.149) and (4.150). The matrix elements p_{11} and p_{12} have the comparatively simpler forms

$$p_{11} = \kappa_0\,\mathrm{ber}(k_1) - k_1^3\,u_1\,\mathrm{bei}'(k_1) + k_1\,u_2\,\mathrm{ber}'(k_1), \tag{4.162}$$

$$p_{12} = \kappa_0\,\mathrm{bei}(k_1) + k_1^3\,u_1\,\mathrm{ber}'(k_1) + k_1\,u_2\,\mathrm{bei}'(k_1). \tag{4.163}$$

Solving (4.160) and (4.161) for C_1 and C_2, we obtain

$$C_1 = \frac{1}{|p|}\left\{p_{22}\,(1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right) - p_{12}\left[(\beta + \beta_\nu)\left(\tau - \frac{q}{2\kappa_o}\right) - \mu\right]\right\}, \tag{4.164}$$

$$C_2 = -\frac{1}{|p|}\left\{p_{21}\,(1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right) - p_{11}\left[(\beta + \beta_\nu)\left(\tau - \frac{q}{2\kappa_o}\right) - \mu\right]\right\}, \tag{4.165}$$

where $|p| \equiv p_{11}p_{22} - p_{12}p_{21}$ is the determinant of the new coefficient matrix.

Using the same model described for the free edge problem, Figures 8 and 9 compare our pinned edge solutions for the axial and radial displacements with the geometrically linear FE solutions.

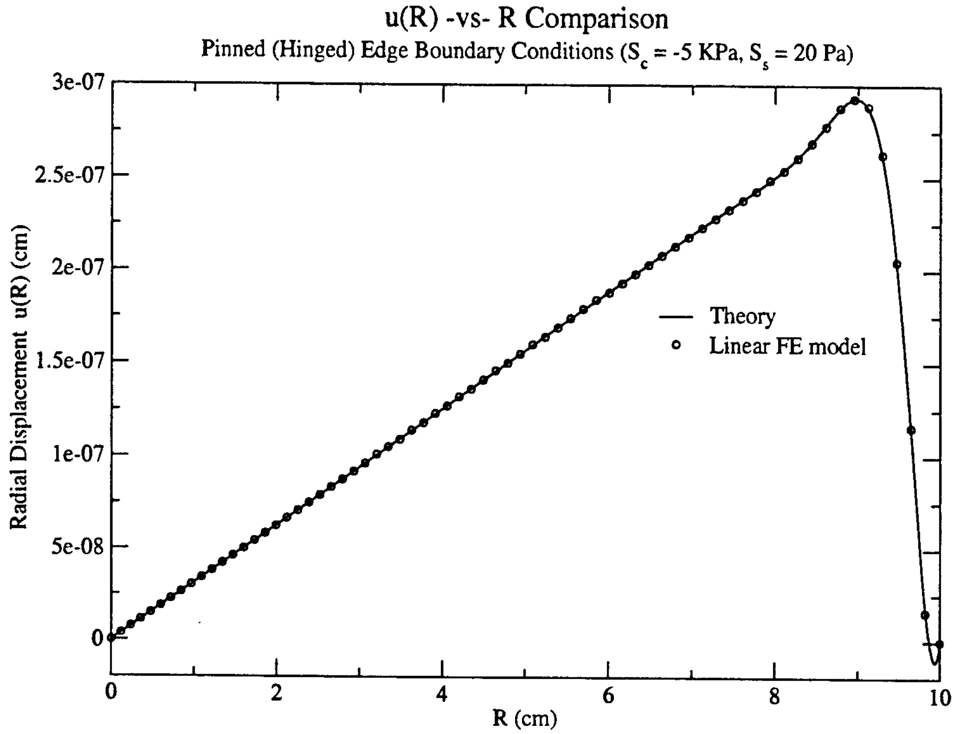

Figure 8: Comparison of theory to geometrically linear FE u-displacement results (pinned edge).

4.2.4 Rigidly Clamped Edge, and Coating Stress Prescriptions for Maintaining an Initially Parabolic Shape

A near net-shape coated membrane used as the primary mirror of a telescope will likely be attached to a rigid circular boundary, hence the boundary conditions to be satisfied for such applications are those of a clamped edge. The clamped edge boundary conditions are $w(a) = 0$, $w_{,R}(a) = 0$, and $u(a) = 0$ or, in terms of ρ, $\widehat{w}(1) = 0$, $\widehat{w}_{,\rho}(1) = 0$, and $\widehat{u}(1) = 0$. The first of the boundary conditions leads, as in the previous problem, to the same forms of the solutions given in (4.158) and (4.120). The coefficients C_1 and C_2 in this case, however, must now be solutions of the two boundary conditions $\widehat{u}(1) = 0$ and $\widehat{w}_{,\rho}(1) = 0$, which can be written as

$$c_{11}C_1 + c_{12}C_2 - (1 - \nu_A)\left(\tau - \frac{q}{2\kappa_o}\right) = 0, \tag{4.166}$$

$$C_1 \mathrm{ber}'(k_1) + C_2 \mathrm{bei}'(k_1) = 0, \tag{4.167}$$

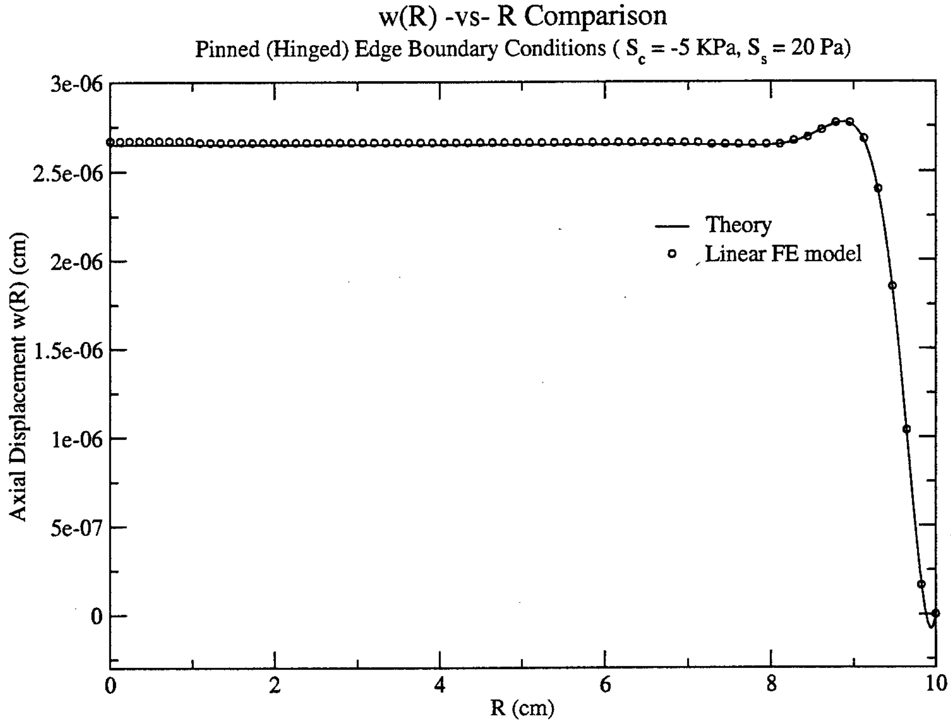

Figure 9: Comparison of theory to geometrically linear FE w-displacement results (pinned edge).

where (4.167) follows from (4.104), and $c_{11} = p_{11}$, $c_{12} = p_{12}$, where p_{11} and p_{12} are given by (4.162) and (4.163), respectively. Equations (4.166) and (4.167) are easily solved, yielding

$$C_1 = \quad (1 - \nu_A) \left(\tau - \frac{q}{2\kappa_o} \right) \left[\frac{\text{bei}'(k_1)}{c_{11}\text{bei}'(k_1) - c_{12}\text{ber}'(k_1)} \right], \tag{4.168}$$

$$C_2 = -(1 - \nu_A) \left(\tau - \frac{q}{2\kappa_o} \right) \left[\frac{\text{ber}'(k_1)}{c_{11}\text{bei}'(k_1) - c_{12}\text{ber}'(k_1)} \right]. \tag{4.169}$$

In this case, both the radial and axial displacement solutions contain the common load factor

$$\tau_L \equiv \tau - \frac{q}{2\kappa_o}, \tag{4.170}$$

which is the *only* occurrence of the intrinsic stresses in the solutions. By choosing the parameters in this factor appropriately, i.e., such that

$$\tau_L \equiv \tau - \frac{q}{2\kappa_o} = 0, \quad \text{or equivalently,} \quad N_0 \equiv \mathcal{N} - \frac{(p + \gamma_0\, g)\, a}{2\kappa_0} = 0, \tag{4.171}$$

it should be possible to achieve a state of *no deformation* from the initial parabolic shape. For example, if there is no pressure difference then one should be able to adjust the coating stress (or one of the coating stresses of a multilayer coating) to satisfy $\mathcal{N} - (\gamma_0\, g/2\kappa_0)\, a = 0$. In a $0g$ environment, this condition reduces to $\mathcal{N} = 0$, which for a single coating is equivalent to having a coating stress given by the simple prescription

$$S_c = -\frac{h_s}{h_c} S_s\,. \tag{4.172}$$

Equation (4.171), and its special case (4.172), define what we refer to as "on-design" prescriptions for a coating stress that will *maintain* the initially parabolic shape of a coated membrane after removal from the mold upon which it was cast and coated. Note that this simple prescription holds only for clamped edge boundary conditions. If the coating stress is less than the on-design value, we say that the membrane is *undercompensated*. On the other hand, if the coating stress is greater than the on-design value, we say that the membrane is *overcompensated*.

The clamped-edge solutions are perhaps the most important for the analysis of near net-shape coated membranes used as optical quality reflectors. For this reason, we have carried out a more extensive comparison with finite element models than for the previous two boundary value problems. The details of this work were presented at the 43rd AIAA Structural, Structural Dynamics, and Materials Conference in April 2002 [20], and accepted for publication in 2003 [21]. Here, we reproduce comparisons of our geometrically linear theory to both geometrically linear and nonlinear finite element results.

The model in this case is a 10 m diameter ($a = 5$ m), $f/2$ coated membrane, in a $0g$ environment. The material and geometrical properties of the coating and membrane are the same as the last two boundary value problems, viz., $E_c = 44$ GPa, $E_s = 2.2$ GPa, $\nu_c = \nu_s = 0.4$, coating thickness $h_c = 1\,\mu$m, and membrane thickness $h_s = 20\,\mu$m. We have somewhat arbitrarily specified a membrane CTE mismatch stress of 11 MPa, which corresponds to an on-design coating stress of -220 MPa, calculated from equation (4.172).

In Figures (10)-(13) we show the predicted effects on the displacement components of *overcompensating* the membrane stress by either 1%, shown in Figures (10) and (11), or 10%, shown in Figures (12) and (13). An edge effect beginning some 10 to 20 cm from the edge is observed in both the theoretical and FE results. The graphs of radial displacement pass through the origin and are linear until the onset of this edge effect, while the axial displacement curves are quite flat until the edge effect occurs. To enhance the detail of these edge effects, we have shown in the Figures only the *final meter* before reaching the edge. As in the previous cases considered, the agreement between theoretical and geometrically linear FE results is excellent.

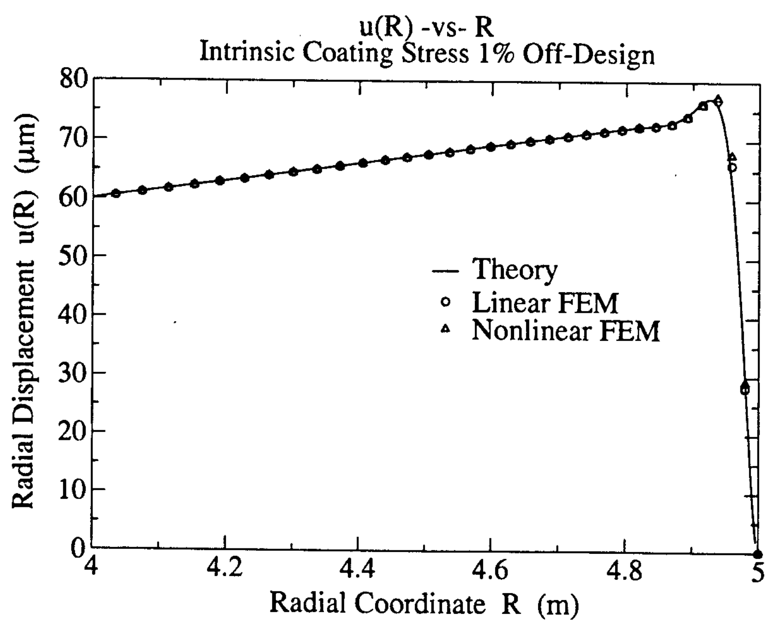

Figure 10: Comparison of theory and finite element results for radial displacement $u(R)$ when the coating stress is 1% off-design.

However, in Figure 11 the geometrically nonlinear axial displacement is roughly 97% of the theoretical prediction for the 1% off-design case, while in Figure 13 the nonlinear prediction is only 72% of the theoretical

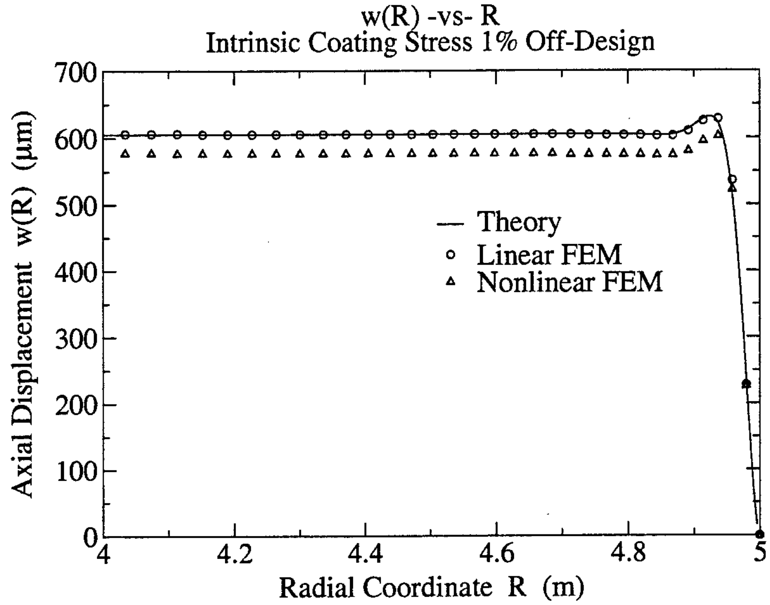

Figure 11: Comparison of theory and finite element results for axial displacement $w(R)$ when the coating stress is 1% off-design.

one. This tendency for the theoretical result (and linear FE results) to *overestimate* the axial displacement is made evident in Figure 14, which indicates that for this example geometrical nonlinearities become important when the coating stress is more than 2% off-design, while the theory is fairly accurate when off-design by less than 2%.

4.3 General Solution for an Initially Flat Laminate

The solution procedure of §4.2 included several steps involving division by the dimensionless parameter κ_0 defined by (4.64). For an initially *flat* laminate, requiring $\kappa = 0$ hence $\kappa_0 = 0$, these divisions cannot be made. The solutions for a flat laminate presumably follow from those of the parabolic laminate in the limit $\kappa_0 \to 0$ (i.e., the focal length $f \to \infty$), but we prefer to return to equations (4.66) and (4.68), prior to any divisions by κ_0, and solve them anew. Setting $\kappa_0 = 0$ in these two equations yields

$$[x_r + x_\theta - \beta_\nu (\kappa_r + \kappa_\theta)]_{,\rho} = 0,$$

and

$$-\Delta (\kappa_r + \kappa_\theta)_{,\rho} + q \frac{\rho}{2} = 0,$$

each of which can be immediately integrated to obtain

$$x_r + x_\theta - \beta_\nu (\kappa_r + \kappa_\theta) = c_1, \tag{4.173}$$

and

$$\kappa_r + \kappa_\theta = c_2 + \frac{q}{\Delta} \frac{\rho^2}{4}, \tag{4.174}$$

where c_1 and c_2 are arbitrary integration constants. Using (4.77), equation (4.174) can be written as

$$\widehat{w}_{,\rho\rho} + \frac{\widehat{w}_{,\rho}}{\rho} = c_2 + \frac{q}{\Delta} \frac{\rho^2}{4}. \tag{4.175}$$

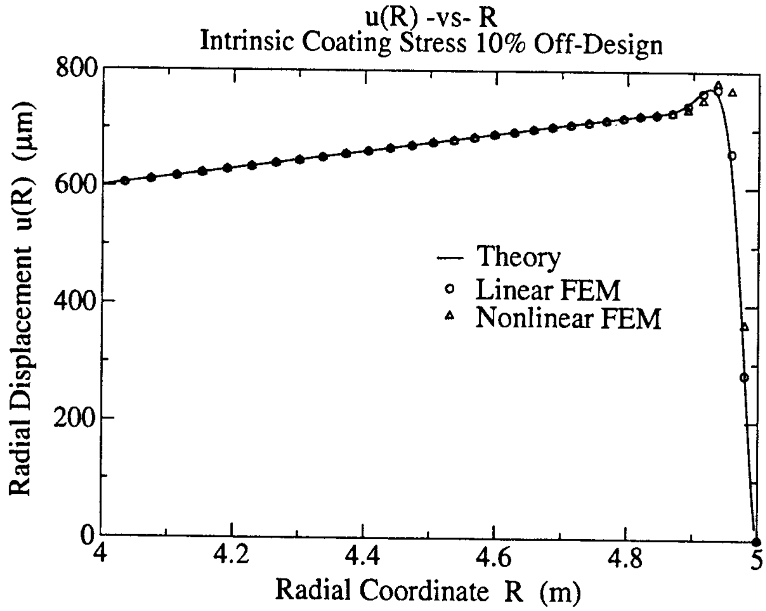

Figure 12: Comparison of theory and finite element results for radial displacement $u(R)$ when the coating stress is 10% off-design.

Multiplying (4.175) through by ρ yields

$$\rho\, \widehat{w}_{,\rho\rho} + \widehat{w}_{,\rho} \equiv \left(\rho\, \widehat{w}_{,\rho}\right)_{,\rho} = c_2\, \rho + \frac{q}{\Delta}\frac{\rho^3}{4},$$

which can be integrated twice to obtain the general solution for $\widehat{w}(\rho)$:

$$\widehat{w}(\rho) = c_4 + c_3 \ln \rho + c_2 \frac{\rho^2}{4} + \frac{q}{\Delta}\frac{\rho^4}{64}, \tag{4.176}$$

where c_3 and c_4 are arbitrary integration constants. We must set $c_3 = 0$ in order for the solution to be regular at $\rho = 0$, hence the general solution regular at the origin is

$$\widehat{w}(\rho) = c_4 + c_2 \frac{\rho^2}{4} + \frac{q}{\Delta}\frac{\rho^4}{64}. \tag{4.177}$$

Returning to equation (4.173), we use equations (5.39) and (5.40) to replace x_r and x_θ, yielding

$$x_r + x_\theta = \frac{1}{1-\nu_A}\left(\epsilon^0_{RR} + \epsilon^0_{\Theta\Theta}\right) + (b + b_\nu)\left(\kappa_r + \kappa_\theta\right) = c_1 + \beta_\nu\left(\kappa_r + \kappa_\theta\right),$$

from which

$$\epsilon^0_{RR} + \epsilon^0_{\Theta\Theta} = c_1\left(1-\nu_A\right) - \left(1-\nu_A^2\right) b \left(\kappa_r + \kappa_\theta\right), \tag{4.178}$$

where we used the definition (4.50) to replace β_ν. Equation (4.174) can be used to eliminate $(\kappa_r + \kappa_\theta)$ in (4.178), yielding

$$\epsilon^0_{RR} + \epsilon^0_{\Theta\Theta} = c_1\left(1-\nu_A\right) - \left(1-\nu_A^2\right) q \frac{b}{\Delta}\frac{\rho^2}{4}. \tag{4.179}$$

Substituting in (4.179) for the strain components from (4.39) (with $\widehat{\Gamma}_{,\rho} = -\kappa_0\, \rho = 0$), we obtain the following differential equation for $\widehat{u}(\rho)$:

$$\widehat{u}_{,\rho} + \frac{\widehat{u}}{\rho} = c_1\left(1-\nu_A\right) - \left(1-\nu_A^2\right) q \frac{b}{\Delta}\frac{\rho^2}{4}.$$

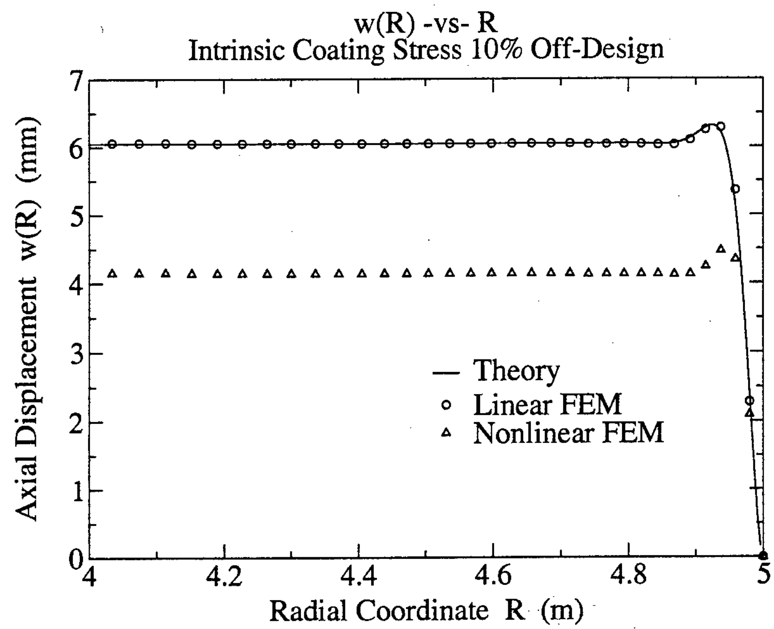

Figure 13: Comparison of theory and finite element results for axial displacement $w(R)$ when the coating stress is 10% off-design.

Multiplying through by ρ and integrating yields the general solution

$$\widehat{u}(\rho) = \frac{c_5}{\rho} + c_1 \, (1-\nu_A) \, \frac{\rho}{2} - \left(1-\nu_A^2\right) q \, \frac{b}{\Delta} \frac{\rho^3}{16}, \tag{4.180}$$

where c_5 is a final integration constant. We must set $c_5 = 0$ to obtain a solution regular at $\rho = 0$:

$$\widehat{u}(\rho) = c_1 \, (1-\nu_A) \, \frac{\rho}{2} - \left(1-\nu_A^2\right) q \, \frac{b}{\Delta} \frac{\rho^3}{16}. \tag{4.181}$$

Restoring the dimensional variables and constants in equations (4.181) and (4.177), we write the general solutions for $u(R)$ and $w(R)$ regular at the origin as

$$u(R) = \epsilon^o R - q_u R^3, \qquad q_u \equiv \frac{B\,(p + \gamma_0\, g)}{16\,(AD - B^2)}, \tag{4.182}$$

and

$$w(R) = w_0 - \frac{k}{2} R^2 + q_w R^4, \qquad q_w \equiv \frac{A\,(p + \gamma_0\, g)}{64\,(AD - B^2)}, \tag{4.183}$$

where we have introduced new arbitrary constants ϵ^o, w_0 and k, as well as coefficients q_u and q_w containing the pressure and gravity loads. The three arbitrary constants ϵ^o, w_0 and k must be determined by applying boundary conditions.

4.3.1 Flat Pressurized Laminate, Clamped at the Edge

For a flat laminate under pressure, the boundary conditions insuring that the edge $R = a$ of the laminate remains fixed are $u(a) = 0$, $w(a) = 0$, and $w'(a) = 0$. Applying the first of these to (4.182) yields an expression for ϵ^0:

$$\epsilon^o = q_u a^2,$$

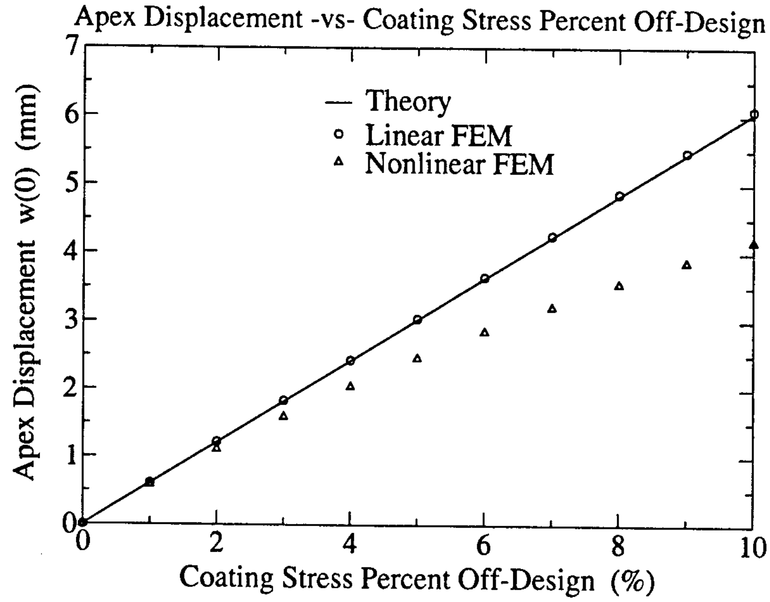

Figure 14: Comparison of theory and finite element results for apex displacement $w(0)$ as a function of the percent that coating stress is off-design.

hence the clamped-edge solution for the radial displacement is

$$u(R) = q_u R\left(a^2 - R^2\right) = \frac{B\,(p + \gamma_0\, g)}{16\,(AD - B^2)} R\left(a^2 - R^2\right). \tag{4.184}$$

The first boundary condition for the axial displacement yields from (4.183) the following expression for the apex displacement w_0:

$$w_0 = \frac{k}{2} a^2 - q_w\, a^4, \tag{4.185}$$

which brings the solution to the form

$$w(R) = \frac{k}{2}\left(a^2 - R^2\right) - q_w\left(a^4 - R^4\right). \tag{4.186}$$

Setting the derivative of this expression, evaluated at $R = a$, to zero yields

$$w'(a) = -k\,a + 4\,q_w\,a^3 = 0,$$

from which

$$k = 4\,q_w\,a^2 \tag{4.187}$$

Substituting this result in (4.185) then gives

$$w_0 = q_w\, a^4 = \frac{A\,(p + \gamma_0\, g)}{64\,(AD - B^2)}\, a^4. \tag{4.188}$$

Using k from (4.187) in (4.186), the clamped-edge solution for the axial displacement reduces to

$$w(R) = q_w\left(a^2 - R^2\right)^2 = \frac{A\,(p + \gamma_0\, g)}{64\,(AD - B^2)}\left(a^2 - R^2\right)^2. \tag{4.189}$$

We note that for a *single* material, the coefficient B appearing in (4.189) and defined in equation (1.3) is zero. The axial displacement solution in that case reduces to the one given in equation (62), p. 55 of [22] (if the effects of gravity are ignored). The general effect of the multilayers is to *reduce* the composite bending stiffness D by an amount B^2/A. It is also worth remarking that these solutions have *no dependence* on the residual stress loads S_i. Thus, in the absence of a pressure load, and if gravity is ignored, then $q_u = 0$ and $q_w = 0$ and *both displacements vanish* for clamped-edge boundary conditions regardless of the residual stress levels.

4.3.2 Unpressurized Laminate with Free Edge: Generalized Stoney Formula

We suppose now that there is no pressure load, and that the effects of gravity can be neglected. Then $q_w = 0$ and $q_u = 0$ in the general solutions (4.182) and (4.183) for the displacements, reducing them to

$$u(R) = \epsilon^o R, \tag{4.190}$$

and

$$w(R) = w_0 - \frac{k}{2} R^2. \tag{4.191}$$

From these solutions and the definitions (4.18) and (4.19) with $\Gamma_{,R} = 0$, we find immediately that

$$\epsilon^0_{RR} = \epsilon^0_{\Theta\Theta} = \epsilon^o, \qquad k_{RR} = k_{\Theta\Theta} = k, \tag{4.192}$$

hence the stress resultants and couples (4.28)-(4.31) reduce to

$$N_R = N_\Theta \equiv N = \mathcal{N} + (A + A_\nu)\, \epsilon^o + (B + B_\nu)\, k, \tag{4.193}$$

and

$$M_R = M_\Theta \equiv M = -\mathcal{M} - (B + B_\nu)\, \epsilon^o - (D + D_\nu)\, k, \tag{4.194}$$

where N and M are *constants*. We note that from the definitions (1.2)–(1.6), the summed coefficients appearing in (4.193) and (4.194) take the forms

$$A + A_\nu = \sum_{i=1}^{N} h_i \mathcal{B}_i, \tag{4.195}$$

$$B + B_\nu = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{k=i+1}^{N} h_i h_k \left(\mathcal{B}_i - \mathcal{B}_k\right), \tag{4.196}$$

$$D + D_\nu = \frac{1}{12} \sum_{i=1}^{N} h_i \mathcal{B}_i \left[h_i^2 + 3\,(h - h_i)^2 - 12\xi_{i-1}\,(h - \xi_i)\right], \tag{4.197}$$

where we have introduced

$$\mathcal{B}_i \equiv Q_i\,(1 + \nu_i) = \frac{E_i}{1 - \nu_i}, \tag{4.198}$$

and made use of the first definition in (1.8), i.e.,

$$Q_i = \frac{E_i}{1 - \nu_i^2}, \tag{4.199}$$

to obtain the last equality of (4.198). As mentioned earlier in §3.3, the quantities $\mathcal{Q}_i$ and $\mathcal{B}_i$ are, somewhat confusingly, *both* referred to in the literature as the *biaxial modulus* of the material in layer i. Henceforth, we reserve the name biaxial modulus for $\mathcal{B}_i$, defined in (4.198). Note that the ratio of biaxial moduli occurred earlier in the condition (4.155) for no axial deformation of a free-edge membrane simply supported at its center.

We now consider a coated membrane laminate with free-edge boundary conditions $N_R(a) = 0$ and $M_R(a) = 0$. These two conditions allow the determination of the arbitrary constants ϵ^o and k appearing in the general solutions (4.190) and (4.191). In fact, we have found in equations (4.193) and (4.194) that both N_R and M_R are constants for this problem, so that ϵ^o and k are simply the algebraic solutions of the linear system in these two unknowns obtained by setting $N_R(a) = N = 0$ and $M_R(a) = M = 0$:

$$0 = \mathcal{N} + (A + A_\nu)\,\epsilon^o + (B + B_\nu)\,k, \tag{4.200}$$

$$0 = -\mathcal{M} - (B + B_\nu)\,\epsilon^o - (D + D_\nu)\,k. \tag{4.201}$$

The solutions are easily found to be

$$\epsilon^o = \frac{-(D + D_\nu)\,\mathcal{N} + (B + B_\nu)\,\mathcal{M}}{(A + A_\nu)(D + D_\nu) - (B + B_\nu)^2}, \tag{4.202}$$

$$k = \frac{(B + B_\nu)\,\mathcal{N} - (A + A_\nu)\,\mathcal{M}}{(A + A_\nu)(D + D_\nu) - (B + B_\nu)^2}. \tag{4.203}$$

This expression for k represents the generalization to a multilayer coated substrate, with arbitrary layer thicknesses, of an important result due originally to G. G. Stoney [23]. Note that, as discussed in §1 following equation (1.20), k is the negative vertex curvature of the paraboloid defined in this case by (4.191). When applied to a substrate with a single coating, it reduces to a result discussed in [24, pp. 140–143], and can also be found (in various forms) in several recent publications [25, 26, 27, 15, 28].

The remaining integration constant $w_0 = w(0)$ is the axial displacement of the center $R = 0$. We assume that there is at least one value of R, say R_0, at which the coated membrane is simply supported, meaning that $w(R_0) = 0$, hence $w_0 = kR_0^2/2$. For support at the center, $R_0 = 0$, we have $w_0 = 0$, and

$$w_{ssc}(R) = -\frac{k}{2}\,R^2, \qquad \text{simply supported at the center,} \tag{4.204}$$

and for support at the edge, $R_0 = a$, we have $w_0 = ka^2/2$, hence

$$w_{sse}(R) = \frac{k}{2}\,\left(a^2 - R^2\right), \qquad \text{simply supported at the edge.} \tag{4.205}$$

In either case, $w'(R) = -kR$ and $w''(R) = -k$, so if $k > 0$ the axial displacement will have its *maximum* value of $w_{sse}(0) = ka^2/2$ at the center, while for $k < 0$ it will have its *minimum* value of $w_{ssc}(0) = 0$ at the center (and the positive value $w_{ssc}(a) = -ka^2/2$ at its edge). The maximum axial deflection, which will occur either at the center or at the edge, is thus

$$w_{max} = \frac{|k|}{2}\,a^2. \tag{4.206}$$

Note that under these boundary conditions the requirements for no displacement of the initially flat coated membrane are $\epsilon^o = 0$ and $k = 0$, which imply $\mathcal{N} = 0$ and $\mathcal{M} = 0$, the same conditions found in (4.154) for no displacement of an initially parabolic membrane satisfying the same boundary conditions.

These results can be applied to the design of multilayer coatings that will compensate for curvature induced by residual stresses in the coatings. For example, Cao, et al [29], discuss such compensation techniques for reducing or eliminating unwanted curvature due to intrinsic coating stresses in high-reflectance thin-film micromirrors, and Liu and Talghader [30] discuss compensating for micromirror curvature due to thermal stresses caused by CTE mismatches between layers. In both papers an extension [31] to multilayer stacks of Timoshenko's [32] one-dimensional analysis of bimetal thermostats is used for a preliminary analysis of the residual stress effects, replacing each instance of a modulus E by the biaxial modulus $E/(1-\nu)$ (but continuing to denote this biaxial modulus by the same symbol E). For comparison with Figure 3 of Reference [29] we have plotted in Figure 15, below, the absolute value $|k|$ of our vertex curvature (4.203) as a function of the thickness h_1 (in nanometers) of an outer dielectric coating of SiO_2, placed over a gold coating of thickness $h_2 = 0.5\,\mu$m on a polysilicon membrane substrate of thickness $h_3 = 3.5\,\mu$m. We have used their data for the various layers, viz., $E_1 = 80\,$GPa, $S_1 = 100\,$MPa for the SiO_2 outer coating, $E_2 = 80\,$GPa, $S_2 = -100\,$MPa for the interior gold coating, and $E_3 = 125\,$GPa, $S_3 = 0$ for the polysilicon substrate. The curvature corresponding to a displacement from an initially flat surface of $w_{max} = |\lambda/10|$ defines a "flatness

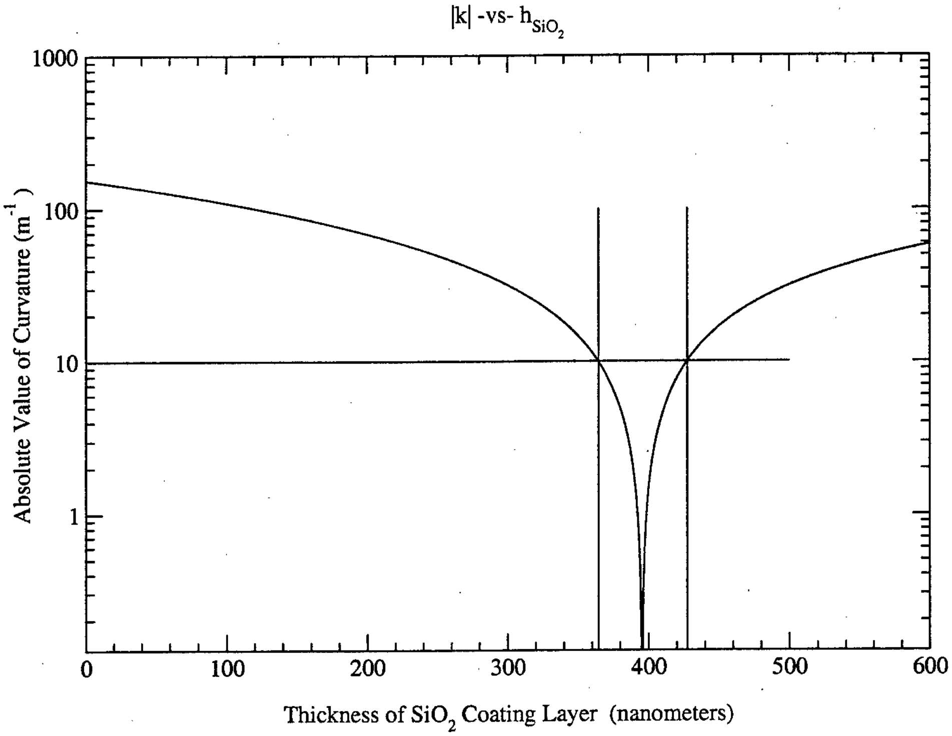

Figure 15: Absolute value $|k|$ as a function of thickness of outer SiO_2 dielectric coating layer.

window" for the allowable thickness variation of the SiO_2 coating. This curvature can be found from (4.206), i.e.,

$$|k| = 2\frac{w_{max}}{a^2} = \frac{\lambda}{5a^2}, \tag{4.207}$$

which requires knowledge of the specimen size. In Reference [29] the specimens were square with sides of length $200\,\mu$m, corresponding roughly to circular specimens of radius $a = 100\,\mu$m. The curvature corresponding to this radius is

$$|k| = \frac{\lambda}{5} \times 10^8, \tag{4.208}$$

so that for visible wavelengths on the order of 500 nm, we have

$$|k| \approx 10, \tag{4.209}$$

corresponding to the line $|k| = 10$ in Figure 15. We observe that for this particular case our flatness window is about 63 nm, compared to the much narrower 20 nm flatness window shown in Figure 3 of [29]. Based on correspondence with Professor Talghader, it is believed that the discrepancy is due to either a typographical error in the data reported for Figure 3, or an error in the simulator being used at the time of publication. A rerun of their current simulator with the data used here shows excellent agreement with the result reported in our Figure 15. It should be noted that the thickness required to flatten the mirror is generally not the same as that which would zero out the *radial* expansion or contraction given by (4.190) and (4.202). In Figure 16 we show the variation of radial edge deflection $u(a) = \epsilon^o a$ in nanometers as the SiO_2 thickness increases, using the same data as that of Figure 15. The flatness window corresponds to a maximum radial expansion of about 3 nm, and a minimum of 1.5 nm.

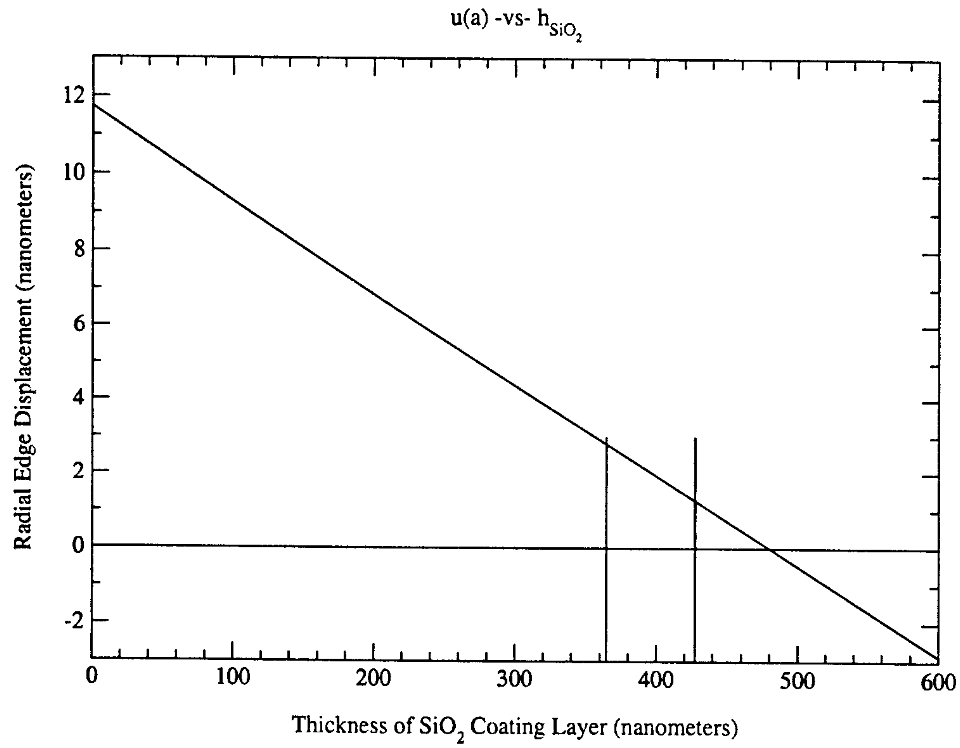

Figure 16: Radial edge displacement $u(a)$ as a function of thickness of outer SiO_2 dielectric coating layer.

5 Geometrically Nonlinear Shell Laminate Problems

In this final Section we present the geometrically nonlinear theory of a coated membrane shell. Such problems are notoriously resistant to analysis in terms of known elementary or special functions. We specialize immediately to the axially symmetric and initially flat subset of these problems. Our hope was to generalize to stress-coated multilayered plates some of the earlier work [33, 34] on plates of a single material with no residual stress. In the work cited, Chia [33, pp. 112-115] applied perturbation theory to find approximate solutions of the plate problem, while Way [34] attacked the problem with power series expansions, a technique that had been used much earlier by Hencky [7] in solving the equations of membrane theory. As we shall see, the existence of residual stresses and the appearance of additional terms due to the multilayer structure, greatly complicate the solution by either perturbation series or power series methods. Our efforts to this point have been for the most part unsuccessful, and we must be content to simply record what has been accomplished in the hope that it may be useful as a reference point for others interested in these types of problems.

As in §4, the leading order displacement components for a nonlinear shell laminate have the Kirchhoff-Love forms given in equation (8.31.I) of Reference [1]:

$$U_Z = w, \qquad U_R = u - Z\, w_{,R}, \qquad U_\Theta = v - Z\,\frac{w_{,\Theta}}{R}, \tag{5.1}$$

where u, v, and w are functions of R and Θ only. The strain components given in equations (8.35.I)–(8.40.I), however, contain nonlinear terms in the displacement component derivatives:

$$\epsilon_{R\Theta} = \frac{1}{2}\left[v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} + \frac{w_{,R} w_{,\Theta}}{R} - 2Z\left(\frac{w_{,R\Theta}}{R} - \frac{w_{,\Theta}}{R^2}\right)\right] \equiv \epsilon^0_{R\Theta} - Z\, k_{R\Theta}, \tag{5.2}$$

$$\epsilon_{RR} = u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2} w_{,R}^2 - Z\, w_{,RR} \equiv \epsilon^0_{RR} - Z\, k_{RR}, \tag{5.3}$$

$$\epsilon_{\Theta\Theta} = \frac{v_{,\Theta} + u}{R} + \frac{w_{,\Theta}^2}{2R^2} - Z\left(\frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2}\right) \equiv \epsilon^0_{\Theta\Theta} - Z\, k_{\Theta\Theta}, \tag{5.4}$$

where the Z-independent terms of the last three equations are given by

$$\epsilon^0_{R\Theta} \equiv \frac{1}{2}\left(v_{,R} + \frac{u_{,\Theta} - v}{R} + \frac{\Gamma_{,R} w_{,\Theta}}{R} + \frac{w_{,R} w_{,\Theta}}{R}\right), \qquad k_{R\Theta} \equiv -\frac{w_{,R\Theta}}{R} + \frac{w_{,\Theta}}{R^2}, \tag{5.5}$$

$$\epsilon^0_{RR} \equiv u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2} w_{,R}^2, \qquad k_{RR} \equiv w_{,RR}, \tag{5.6}$$

$$\epsilon^0_{\Theta\Theta} \equiv \frac{v_{,\Theta} + u}{R} + \frac{w_{,\Theta}^2}{2R^2}, \qquad k_{\Theta\Theta} \equiv \frac{w_{,R}}{R} + \frac{w_{,\Theta\Theta}}{R^2}, \tag{5.7}$$

The equilibrium equations are given in terms of stress resultants by equations (8.130.I)–(8.132.I):

$$(RN_R)_{,R} - N_\Theta + N_{R\Theta,\Theta} = 0, \tag{5.8}$$

$$\left(R^2 N_{R\Theta}\right)_{,R} + R\, N_{\Theta,\Theta} = 0, \tag{5.9}$$

and

$$\left[R\left(\omega_{,R} N_R + \frac{w_{,\Theta}}{R} N_{R\Theta} + Q_R\right)\right]_{,R} + \left(\omega_{,R} N_{R\Theta} + \frac{w_{,\Theta}}{R} N_\Theta + Q_\Theta\right)_{,\Theta} + (p + \gamma_0\, g)\, R = 0, \tag{5.10}$$

where we recall that $\omega(R) \equiv w(R,\Theta) + \Gamma(R)$. The shear stress resultants Q_R and Q_Θ appearing in the last equilibrium equation can be eliminated in favor of stress couples, which yields the alternative equation (8.140.I) of Reference [1]:

$$\begin{aligned}&\left[R\left(\omega_{,R} N_R + \frac{w_{,\Theta}}{R} N_{R\Theta}\right) + (R\, M_R)_{,R} - M_\Theta + M_{R\Theta,\Theta}\right]_{,R} \\ &\quad + \left[\omega_{,R} N_{R\Theta} + \frac{w_{,\Theta}}{R} N_\Theta + \frac{1}{R^2}\left(R^2 M_{R\Theta}\right)_{,R} + \frac{1}{R} M_{\Theta,\Theta}\right]_{,\Theta} + (p + \gamma_0\, g)\, R = 0.\end{aligned} \tag{5.11}$$

The stress resultants and couples appearing in the equilibrium equations are given by equations (8.141.I)–(8.146.I):

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}, \tag{5.12}$$

$$N_{R\Theta} = A_\Theta\,\epsilon^0_{R\Theta} + B_\Theta\,k_{R\Theta}, \tag{5.13}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta} + B_\nu\,k_{RR} + B\,k_{\Theta\Theta}, \tag{5.14}$$

$$M_R = -\mathcal{M} - B\,\epsilon^0_{RR} - B_\nu\,\epsilon^0_{\Theta\Theta} - D\,k_{RR} - D_\nu\,k_{\Theta\Theta}, \tag{5.15}$$

$$M_{R\Theta} = -B_\Theta\,\epsilon^0_{R\Theta} - D_\Theta\,k_{R\Theta}, \tag{5.16}$$

$$M_\Theta = -\mathcal{M} - B_\nu\,\epsilon^0_{RR} - B\,\epsilon^0_{\Theta\Theta} - D_\nu\,k_{RR} - D\,k_{\Theta\Theta}, \tag{5.17}$$

which are the same as those of the previous Section, except that the strains contain nonlinear terms, according to equations (5.5)–(5.7). The constants appearing here, as elsewhere, are defined in §1.

5.1 Reduction to an Axisymmetric System

The reduction to an axisymmetric system follows along the same lines as §4.1, i.e., we assume that all variables are independent of Θ, and find from equilibrium equation (5.9) that

$$N_{R\Theta} = 0, \text{ for all } R, \tag{5.18}$$

hence from (5.13), since $k_{R\Theta} = 0$ from (5.5) (and A_Θ is, in general, nonzero):

$$\epsilon^0_{R\Theta} = \frac{1}{2}\left(v_{,R} - \frac{v}{R}\right) = 0. \tag{5.19}$$

The last equation can be integrated to give $v(R) = v_0 R$, where v_0 is an arbitrary constant, hence if v vanishes for ***any*** non-zero value of R, then v vanishes for ***all*** R. These results imply from (5.16) that

$$M_{R\Theta} = 0, \text{ for all } R. \tag{5.20}$$

Noting that equation (5.11) can be integrated once (and the integration constant discarded to insure regularity), the system of equations to be solved thereby reduces to the following:

$$(RN_R)_{,R} - N_\Theta = 0, \tag{5.21}$$

$$(R\,M_R)_{,R} - M_\Theta + R\,(w_{,R} + \Gamma_{,R})\,N_R + (p + \gamma_0\,g)\,\frac{R^2}{2} = 0. \tag{5.22}$$

where

$$N_R = \mathcal{N} + A\,\epsilon^0_{RR} + A_\nu\,\epsilon^0_{\Theta\Theta} + B\,k_{RR} + B_\nu\,k_{\Theta\Theta}, \tag{5.23}$$

$$N_\Theta = \mathcal{N} + A_\nu\,\epsilon^0_{RR} + A\,\epsilon^0_{\Theta\Theta} + B_\nu\,k_{RR} + B\,k_{\Theta\Theta}, \tag{5.24}$$

$$M_R = -\mathcal{M} - B\,\epsilon^0_{RR} - B_\nu\,\epsilon^0_{\Theta\Theta} - D\,k_{RR} - D_\nu\,k_{\Theta\Theta}, \tag{5.25}$$

$$M_\Theta = -\mathcal{M} - B_\nu \epsilon^0_{RR} - B \epsilon^0_{\Theta\Theta} - D_\nu k_{RR} - D k_{\Theta\Theta}, \tag{5.26}$$

and

$$\epsilon^0_{RR} = u_{,R} + \Gamma_{,R} w_{,R} + \frac{1}{2} w^2_{,R}, \qquad k_{RR} = w_{,RR}, \tag{5.27}$$

$$\epsilon^0_{\Theta\Theta} = \frac{u}{R}, \qquad k_{\Theta\Theta} = \frac{w_{,R}}{R}. \tag{5.28}$$

As in §4.1, we transform these equations to dimensionless forms by introducing a dimensionless coordinate ρ, dimensionless displacement components $\widehat{u}$ and $\widehat{w}$, and a dimensionless initially curved reference surface $\widehat{\Gamma}$, defined by

$$\rho \equiv \frac{R}{a}, \qquad \widehat{u} \equiv \frac{u}{a}, \qquad \widehat{w} \equiv \frac{w}{a}, \qquad \widehat{\Gamma} \equiv \frac{\Gamma}{a}, \tag{5.29}$$

together with the following dimensionless constants:

$$\nu_A \equiv \frac{A_\nu}{A}, \qquad \overline{E} \equiv \frac{A}{h}\left(1 - \nu_A^2\right), \qquad \tau \equiv \frac{\mathcal{N}}{\overline{E}\, h}, \qquad \mu \equiv \frac{\mathcal{M}}{\overline{E}\, h\, a}, \qquad q \equiv \frac{(p + \gamma_0 g)\, a}{\overline{E}\, h}, \tag{5.30}$$

and four dimensionless dependent variables x_r, x_θ, y_r, and y_θ defined by

$$x_r \equiv \frac{N_R - \mathcal{N}}{\overline{E}\, h}, \qquad x_\theta \equiv \frac{N_\Theta - \mathcal{N}}{\overline{E}\, h}, \qquad y_r \equiv \frac{M_R + \mathcal{M}}{\overline{E}\, h\, a}, \qquad y_\theta \equiv \frac{M_\Theta + \mathcal{M}}{\overline{E}\, h\, a}. \tag{5.31}$$

We also introduce dimensionless curvatures κ_r and κ_θ defined by

$$\kappa_r \equiv a\, k_{RR} = \widehat{w}_{,\rho\rho} \qquad \kappa_\theta \equiv a\, k_{\Theta\Theta} = \frac{\widehat{w}_{,\rho}}{\rho}, \tag{5.32}$$

and four new dimensionless constants

$$b \equiv \frac{B}{\overline{E}\, h\, a}, \qquad b_\nu \equiv \frac{B_\nu}{\overline{E}\, h\, a}, \qquad d \equiv \frac{D}{\overline{E}\, h\, a^2}, \qquad d_\nu \equiv \frac{D_\nu}{\overline{E}\, h\, a^2}, \tag{5.33}$$

noting that the strain components are already dimensionless, and have the following forms in terms of the dimensionless variables defined above:

$$\epsilon^0_{RR} = \widehat{u}_{,\rho} + \widehat{\Gamma}_{,\rho}\, \widehat{w}_{,\rho} + \frac{1}{2} \widehat{w}^2_{,\rho}, \qquad \epsilon^0_{\Theta\Theta} = \frac{\widehat{u}}{\rho}. \tag{5.34}$$

The strain components must satisfy a modified form of the compatibility condition (3.40), viz.,

$$\rho\, \epsilon^0_{\Theta\Theta,\rho} = \epsilon^0_{RR} - \epsilon^0_{\Theta\Theta} - \widehat{\Gamma}_{,\rho}\, \widehat{w}_{,\rho} - \frac{1}{2} \widehat{w}^2_{,\rho}, \tag{5.35}$$

and there is an additional compatibility condition involving the dimensionless curvatures:

$$\rho\, \kappa_{\theta,\rho} = \kappa_r - \kappa_\theta, \tag{5.36}$$

which is easily derived from the definitions (5.32).

Substituting these dimensionless quantities into the equilibrium equations (5.21) and (5.22), we reduce them to

$$x_{r,\rho} + \frac{1}{\rho}\left(x_r - x_\theta \right) = 0, \tag{5.37}$$

and

$$y_{r,\rho} + \frac{1}{\rho}\left(y_r - y_\theta \right) + \left(\widehat{w}_{,\rho} + \widehat{\Gamma}_{,\rho} \right)\left(\tau + x_r \right) + q\, \frac{\rho}{2} = 0, \tag{5.38}$$

respectively. The constitutive relations (5.23)–(5.26) take the forms

$$x_r = \frac{1}{1-\nu_A^2}\left(\epsilon^0_{RR} + \nu_A\,\epsilon^0_{\Theta\Theta}\right) + b\,\kappa_r + b_\nu\,\kappa_\theta, \tag{5.39}$$

$$x_\theta = \frac{1}{1-\nu_A^2}\left(\nu_A\,\epsilon^0_{RR} + \epsilon^0_{\Theta\Theta}\right) + b_\nu\,\kappa_r + b\,\kappa_\theta, \tag{5.40}$$

$$y_r = -b\,\epsilon^0_{RR} - b_\nu\,\epsilon^0_{\Theta\Theta} - d\,\kappa_r - d_\nu\,\kappa_\theta, \tag{5.41}$$

$$y_\theta = -b_\nu\,\epsilon^0_{RR} - b\,\epsilon^0_{\Theta\Theta} - d_\nu\,\kappa_r - d\,\kappa_\theta. \tag{5.42}$$

Equations (5.39) and (5.40) are easily solved for ϵ^0_{RR} and $\epsilon^0_{\Theta\Theta}$ in terms of the x's and κ's:

$$\epsilon^0_{RR} = x_r - \nu_A\,x_\theta - \beta\,\kappa_r - \beta_\nu\,\kappa_\theta, \tag{5.43}$$

$$\epsilon^0_{\Theta\Theta} = x_\theta - \nu_A\,x_r - \beta_\nu\,\kappa_r - \beta\,\kappa_\theta, \tag{5.44}$$

where we recall the definitions first given in (4.50):

$$\beta \equiv b - \nu_A\,b_\nu, \qquad \beta_\nu \equiv b_\nu - \nu_A\,b. \tag{5.45}$$

We note here that the dimensionless radial displacement component can be expressed in terms of x_r and x_θ, and κ_r and κ_θ (which themselves depend only on derivatives of $\widehat{w}$), by combining (5.34) and (5.44) to obtain

$$\widehat{u}(\rho) = \rho\left(x_\theta - \nu_A\,x_r - \beta_\nu\,\kappa_r - \beta\,\kappa_\theta\right). \tag{5.46}$$

Taking the derivative with respect to ρ of (5.44) and substituting the result, together with (5.43) and (5.44), in the ϵ-compatibility relation (5.35), we obtain

$$\rho\left(x_{\theta,\rho} - \nu_A\,x_{r,\rho} - \beta_\nu\,\kappa_{r,\rho} - \beta\,\kappa_{\theta,\rho}\right) = (1+\nu_A)(x_r - x_\theta) - (\beta - \beta_\nu)(\kappa_r - \kappa_\theta) - \widehat{\Gamma}_{,\rho}\,\widehat{w}_{,\rho} - \frac{1}{2}\widehat{w}^2_{,\rho}.$$

In this expression, we use (5.36) to replace $(\kappa_r - \kappa_\theta)$, and (5.37) to replace $(x_r - x_\theta)$, which yields the following result for the compatibility condition:

$$\rho\left[x_r + x_\theta - \beta_\nu\left(\kappa_r + \kappa_\theta\right)\right]_{,\rho} = -\widehat{\Gamma}_{,\rho}\,\widehat{w}_{,\rho} - \frac{1}{2}\widehat{w}^2_{,\rho}. \tag{5.47}$$

Next, we substitute (5.43) and (5.44) in (5.41) and (5.42) to obtain

$$y_r = -\beta\,x_r - \beta_\nu\,x_\theta - \delta\,\kappa_r - \delta_\nu\,\kappa_\theta, \tag{5.48}$$

$$y_\theta = -\beta_\nu\,x_r - \beta\,x_\theta - \delta_\nu\,\kappa_r - \delta\,\kappa_\theta, \tag{5.49}$$

where we introduced another pair of constants:

$$\delta \equiv d - b\beta - b_\nu\beta_\nu, \qquad \delta_\nu \equiv d_\nu - b\beta_\nu - b_\nu\beta. \tag{5.50}$$

The derivative of y_r with respect to ρ yields from (5.48):

$$y_{r,\rho} = -\beta\,x_{r,\rho} - \beta_\nu\,x_{\theta,\rho} - \delta\,\kappa_{r,\rho} - \delta_\nu\,\kappa_{\theta,\rho}. \tag{5.51}$$

From the last three equations we obtain

$$y_{r,\rho} + \frac{1}{\rho}(y_r - y_\theta) = -\beta\, x_{r,\rho} - \beta_\nu\, x_{\theta,\rho} - \delta\, \kappa_{r,\rho} - \delta_\nu\, \kappa_{\theta,\rho}$$
$$- \frac{1}{\rho}\left[(\beta - \beta_\nu)(x_r - x_\theta) - (\delta - \delta_\nu)(\kappa_r - \kappa_\theta)\right].$$

In this expression, we again use (5.36) to replace $(\kappa_r - \kappa_\theta)$, and (5.37) to replace $(x_r - x_\theta)$, which simplifies it to

$$y_{r,\rho} + \frac{1}{\rho}(y_r - y_\theta) = -\beta_\nu\, (x_r + x_\theta)_{,\rho} - \delta\, (\kappa_r + \kappa_\theta)_{,\rho}. \tag{5.52}$$

Substitution of (5.52) into (5.38) yields the following form of the axial equilibrium equation:

$$-\beta_\nu\, (x_r + x_\theta)_{,\rho} - \delta\, (\kappa_r + \kappa_\theta)_{,\rho} + \left(\widehat{w}_{,\rho} + \widehat{\Gamma}_{,\rho}\right)(\tau + x_r) + q\,\frac{\rho}{2} = 0. \tag{5.53}$$

Equation (5.37) can be used to eliminate x_θ in (5.47) and (5.53), and the problem we are left with is to solve the coupled differential equations (5.47) and (5.53) for $\widehat{w}$ and x_r. The radial displacement $\widehat{u}$ is then determined from (5.46).

5.2 Approximate Solutions for Initially Flat Laminates Using Perturbation Methods

Here, we apply perturbation methods to an initially flat coated membrane laminate, for which $\widehat{\Gamma}(\rho) = 0$. The problems we consider will be of two distinct types. In the first we are concerned with finding approximate pressure -vs- axial displacement relations for fixed residual stress loads defined by τ. In the second, the pressure and gravity loads defined by q are assumed absent, and we investigate the *compressive* intrinsic coating stress contained in τ required to produce buckling, i.e., out-of-plane deflections of the laminate from its initially flat state.

5.2.1 Pressure Versus Axial Displacement Curves

For the first type of problem, we follow Chia [33], assuming series expansions for the dimensionless variables (which, however, are *not the same* as Chia's dimensionless variables) in a perturbation parameter $\omega \equiv \widehat{w}(0)$, the central axial displacement, viz.,

$$\widehat{w}(\rho) = \sum_{n=1} w_n(\rho)\,\omega^n = w_1(\rho)\,\omega + w_2(\rho)\,\omega^2 + w_3(\rho)\,\omega^3 + \ldots. \tag{5.54}$$

$$x_r(\rho) = \sum_{n=1} x_n(\rho)\,\omega^n = x_1(\rho)\,\omega + x_2(\rho)\,\omega^2 + x_3(\rho)\,\omega^3 + \ldots, \tag{5.55}$$

and

$$q = \sum_{n=1} q_n\,\omega^n = q_1\,\omega + q_2\,\omega^2 + q_3\,\omega^3 + \ldots. \tag{5.56}$$

Denoting a derivative with respect to ρ by a "prime", we then have

$$\widehat{w}_{,\rho} = \sum_{n=1} w_n'\,\omega^n, \qquad \widehat{w}_{,\rho\rho} = \sum_{n=1} w_n''\,\omega^n, \qquad x_{r,\rho} = \sum_{n=1} x_n'\,\omega^n. \tag{5.57}$$

The nonlinear terms appearing in equations (5.47) and (5.53) are given to fourth order in ω by

$$(\widehat{w}_{,\rho})^2 = (w_1')^2\,\omega^2 + 2\,w_1'\,w_2'\,\omega^3 + \left[(w_2')^2 + 2\,w_1'\,w_3'\right]\,\omega^4 + \ldots, \tag{5.58}$$

and

$$(\widehat{w}_{,\rho})\,(x_r\,+\,\tau)\;=\;w_1'\,\tau\,\omega+[w_1'\,(x_1+\tau)+w_2'\,\tau)]\;\omega^2+[w_1'\,(x_2+\tau)+w_2'\,(x_1+\tau)+w_3'\,\tau]\;\omega^3$$
$$+\,[w_1'\,(x_3+\tau)+w_2'\,(x_2+\tau)+w_3'\,(x_1+\tau)+w_4'\,\tau]\;\omega^4+\ldots, \tag{5.59}$$

respectively. When these perturbation series are substituted in (5.47) and (5.53), we obtain the following equations as coefficients of the indicated powers of ω for the first four orders:

$$\omega^1:\;\begin{cases}\rho\,(\rho\,x_1'\,+\,2\,x_1)'\,-\,\beta_\nu\,\rho\left(w_1''\,+\,\frac{w_1'}{\rho}\right)'\,=\,0,\\[2ex] \beta_\nu\,(\rho x_1'\,+\,2x_1)'\,+\,\delta\left(w_1''\,+\,\frac{w_1'}{\rho}\right)'\,-\,\frac{1}{2}\,q_1\rho\,-\,w_1'\,\tau\,=\,0,\end{cases} \tag{5.60}$$

$$\omega^2:\;\begin{cases}\rho\,(\rho\,x_2'\,+\,2\,x_2)'\,-\,\beta_\nu\,\rho\left(w_2''\,+\,\frac{w_2'}{\rho}\right)'\,+\,\frac{1}{2}\,(w_1')^2\,=\,0,\\[2ex] \beta_\nu\,(\rho x_2'\,+\,2x_2)'\,+\,\delta\left(w_2''\,+\,\frac{w_2'}{\rho}\right)'\,-\,\frac{1}{2}\,q_2\rho\,-\,w_1'\,(x_1\,+\,\tau)\,-\,w_2'\,\tau\,=\,0,\end{cases} \tag{5.61}$$

$$\omega^3:\;\begin{cases}\rho\,(\rho\,x_3'\,+\,2\,x_3)'\,-\,\beta_\nu\,\rho\left(w_3''\,+\,\frac{w_3'}{\rho}\right)'\,+\,w_1'\,w_2'\,=\,0,\\[2ex] \beta_\nu\,(\rho x_3'\,+\,2\,x_3)'\,+\,\delta\left(w_3''\,+\,\frac{w_3'}{\rho}\right)'\,-\,\frac{1}{2}\,q_3\rho\,-\,w_1'\,(x_2\,+\,\tau)\,-\,w_2'\,(x_1\,+\,\tau)\,-\,w_3'\,\tau\,=\,0,\end{cases} \tag{5.62}$$

$$\omega^4:\;\begin{cases}\rho\,(\rho\,x_4'\,+\,2\,x_4)'\,-\,\beta_\nu\,\rho\left(w_4''\,+\,\frac{w_4'}{\rho}\right)'\,+\,\frac{1}{2}\left[(w_2')^2\,+\,2\,w_1'\,w_3'\right]\,=\,0,\\[2ex] \beta_\nu\,(\rho x_4'\,+\,2\,x_4)'\,+\,\delta\left(w_4''\,+\,\frac{w_4'}{\rho}\right)'\,-\,\frac{1}{2}\,q_4\rho\,-\,w_1'\,(x_3\,+\,\tau)\\ \qquad\qquad -\,w_2'\,(x_2\,+\,\tau)\,-\,w_3'\,(x_1\,+\,\tau)\,-\,w_4'\,\tau\,=\,0.\end{cases} \tag{5.63}$$

The boundary conditions for a rigidly clamped edge at $R=a$, or $\rho=1$, are

$$\widehat{u}(1)\,=\,0,\qquad \widehat{w}(1)\,=\,0,\quad\text{and}\quad \widehat{w}_{,\rho}(1)\,=\,0. \tag{5.64}$$

Recalling equation (5.46) for $\widehat{u}(\rho)$ in terms of x_r, x_θ, and $\widehat{w}$, and using $x_\theta\,=\,\rho\,x_{r,\rho}\,+\,x_r$ from (5.37) to replace x_θ in (5.46), the first boundary condition of (5.64) is equivalent to

$$x_{r,\rho}(1)\,+\,(1\,-\,\nu_A)\,x_r(1)\,-\,\beta_\nu\,\widehat{w}_{,\rho\rho}(1)\,=\,0, \tag{5.65}$$

where we used the third boundary condition to eliminate the term proportional to β. Substitution of the perturbation series (5.54)–(5.57) in these boundary conditions yields the rigidly clamped edge boundary conditions on the perturbation coefficient functions, viz.,

$$x_n'(1)\,+\,(1\,-\,\nu_A)\,x_n(1)\,-\,\beta_\nu\,w_n''(1)\,=\,0,\qquad w_n(1)\,=\,0,\quad\text{and}\quad w_n'(1)\,=\,0. \tag{5.66}$$

In addition, we have the so-called "false" [35, p. 167] boundary conditions on the functions w_n evaluated on axis at $\rho=0$:

$$w_1(0)\,=\,1,\quad\text{and}\quad w_n(0)\,=\,0,\quad\text{for}\quad n\neq 1, \tag{5.67}$$

which follow from the requirement that the apex displacement $\widehat{w}(0)\equiv\omega$ in (5.54).

As our first application (although somewhat of a digression), we consider the specialization of these perturbation equations to true membranes. Comparing equations (3.43) and (3.39) of §3.2 to equations (5.47) and (5.53), respectively (with $\widehat{\Gamma}(\rho) = 0$), we see that the geometrically nonlinear membrane theory follows from the latter equations by setting $\delta = 0$ and $\beta_\nu = 0$. The system (5.60)–(5.63) of perturbation equations thereby reduce to

$$\omega^1 : \begin{cases} \rho\,(\rho\,x_1' + 2\,x_1)' = 0, \\ -\tfrac{1}{2}\,q_1\rho - w_1'\,\tau = 0, \end{cases} \tag{5.68}$$

$$\omega^2 : \begin{cases} \rho\,(\rho\,x_2' + 2\,x_2)' + \tfrac{1}{2}\,(w_1')^2 = 0, \\ -\tfrac{1}{2}\,q_2\rho - w_1'\,(x_1 + \tau) - w_2'\,\tau = 0, \end{cases} \tag{5.69}$$

$$\omega^3 : \begin{cases} \rho\,(\rho\,x_3' + 2\,x_3)' + w_1'\,w_2' = 0, \\ -\tfrac{1}{2}\,q_3\rho - w_1'\,(x_2 + \tau) - w_2'\,(x_1 + \tau) - w_3'\,\tau = 0, \end{cases} \tag{5.70}$$

$$\omega^4 : \begin{cases} \rho\,(\rho\,x_4' + 2\,x_4)' + \tfrac{1}{2}\left[(w_2')^2 + 2\,w_1'\,w_3'\right] = 0, \\ -\tfrac{1}{2}\,q_4\rho - w_1'\,(x_3 + \tau) - w_2'\,(x_2 + \tau) - w_3'\,(x_1 + \tau) - w_4'\,\tau = 0, \end{cases} \tag{5.71}$$

$$\omega^5 : \begin{cases} \rho\,(\rho\,x_5' + 2\,x_5)' + (w_1'\,w_4' + w_2'\,w_3') = 0, \\ -\tfrac{1}{2}\,q_5\rho - w_1'\,(x_4 + \tau) - w_2'\,(x_3 + \tau) - w_3'\,(x_2 + \tau) - w_4'\,(x_1 + \tau) - w_5'\,\tau = 0, \end{cases} \tag{5.72}$$

where we have appended the fifth-order equations. The boundary conditions (5.66) for a rigidly clamped edge reduce to

$$x_n'(1) + (1 - \nu_A)\,x_n(1) = 0, \qquad w_n(1) = 0, \tag{5.73}$$

noting that the boundary condition $\widehat{w}_{,\rho}(1) = 0$ of (5.64) does not apply to a true membrane. The apex displacement conditions (5.67) are unchanged. In the process of solving these equations for the functions $x_i(\rho)$, the following identity is useful:

$$(\rho\,x_i' + 2\,x_i)' \equiv \left[\frac{1}{\rho}\,(\rho^2\,x_i)'\right]'. \tag{5.74}$$

Beginning with the first of equations (5.68), two integrations give the general solution

$$x_1(\rho) = c_1 + \frac{c_2}{\rho^2}, \tag{5.75}$$

where c_1 and c_2 are arbitrary integration constants. We must set $c_2 = 0$ to insure that x_1 is regular at $\rho = 0$. The first boundary condition of (5.73), with $n = 1$, then yields $c_1 = 0$ hence

$$x_1(\rho) = 0, \ \forall\rho. \tag{5.76}$$

The second of equations (5.68) can be integrated to give the general solution

$$w_1(\rho) = -\frac{q_1}{4\,\tau}\,\rho^2 + c_3, \tag{5.77}$$

and the first condition of (5.67) yields $c_3 = 1$, hence

$$w_1(\rho) = 1 - \frac{q_1}{4\tau}\rho^2. \tag{5.78}$$

The second boundary condition of (5.73), with $n = 1$, then yields

$$q_1 = 4\tau, \tag{5.79}$$

and the w_1-solution (5.78) reduces to

$$w_1(\rho) = 1 - \rho^2, \quad \Rightarrow \quad w_1' = -2\rho. \tag{5.80}$$

Substituting for w_1' in the first of equations (5.69), two integrations give the general solution

$$x_2(\rho) = c_1 - \frac{1}{4}\rho^2, \quad \Rightarrow \quad x_2' = -\frac{1}{2}\rho. \tag{5.81}$$

where one integration constant has been set to zero to insure regularity. The first boundary condition of (5.73), with $n = 2$, then yields

$$c_1 = \frac{1}{4}\left(\frac{3-\nu_A}{1-\nu_A}\right), \tag{5.82}$$

which can be substituted in (5.81) to obtain

$$x_2(\rho) = \frac{1}{4}\left(\frac{3-\nu_A}{1-\nu_A} - \rho^2\right). \tag{5.83}$$

Substituting for x_1 and w_1' in the second of equations (5.69), and integrating the result once, yields

$$w_2(\rho) = \left(1 - \frac{q_2}{4\tau}\right)\rho^2 + c_2, \tag{5.84}$$

and the condition (5.67) with $n = 2$ forces $c_2 = 0$. The final boundary condition $w_2(1) = 0$ then gives

$$q_2 = 4\tau, \tag{5.85}$$

which yields from (5.84):

$$w_2(\rho) = 0, \ \forall\rho. \tag{5.86}$$

Continuing, we substitute $w_2' = 0$ from (5.86) in the first of equations (5.69), which leads to the solution

$$x_3(\rho) = 0, \ \forall\rho, \tag{5.87}$$

after applying the first boundary condition of (5.73). Substituting for x_1, x_2, w_1' and w_2' in the second of equations (5.69) yields an equation which integrates to

$$w_3(\rho) = -\frac{1}{8\tau}\rho^4 - \frac{q_3}{4\tau}\rho^2 + \frac{1}{4\tau}\left(\frac{3-\nu_A}{1-\nu_A}\right)\rho^2 + \rho^2, \tag{5.88}$$

after setting the integration constant to zero to satisfy $w_3(0) = 0$. The boundary condition $w_3(1) = 0$ then gives

$$q_3 = 4\tau + \frac{1}{2}\left(\frac{5-\nu_A}{1-\nu_A}\right), \tag{5.89}$$

which yields from (5.88):

$$w_3(\rho) = \frac{1}{8\tau}\rho^2\left(1-\rho^2\right), \quad \Rightarrow \quad w_3' = \frac{1}{4\tau}\rho\left(1-2\rho^2\right). \tag{5.90}$$

Proceeding in the same way for the next two orders we obtain the following solutions:

$$x_4(\rho) = \frac{1}{48\tau}\left[\frac{1+\nu_A}{1-\nu_A} - \rho^2\left(3-2\rho^2\right)\right], \tag{5.91}$$

$$q_4 = 4\tau, \tag{5.92}$$

$$w_4(\rho) = -\frac{1}{8\tau}\rho^2\left(1-\rho^2\right), \quad \Rightarrow \quad w_4' = -\frac{1}{4\tau}\rho\left(1-2\rho^2\right), \tag{5.93}$$

for $n = 4$, and

$$x_5(\rho) = -\frac{1}{48\tau}\left[\frac{1+\nu_A}{1-\nu_A} - \rho^2\left(3-2\rho^2\right)\right], \tag{5.94}$$

$$q_5 = 4\tau + \frac{1}{144\tau}\left(\frac{19+5\nu_A}{1-\nu_A}\right), \tag{5.95}$$

$$w_5(\rho) = -\frac{1}{576\tau^2}\left(\frac{61-25\nu_A}{1-\nu_A}\right)\rho^2 + \frac{1}{64\tau^2}\left(\frac{9-5\nu_A}{1-\nu_A}\right)\rho^4 - \frac{5}{144\tau^2}\rho^6, \tag{5.96}$$

for $n = 5$. Note that $w_4(\rho) = -w_3(\rho)$, and $x_5(\rho) = -x_4(\rho)$. Substituting the solutions q_i, $i = 1, \ldots, 5$ into the perturbation expansion (5.56) for the dimensionless load q, we obtain

$$q = 4\tau\omega\left(1+\omega+\omega^2+\omega^3+\omega^4\right) + \frac{1}{2}\left(\frac{5-\nu_A}{1-\nu_A}\right)\omega^3 + \frac{1}{144\tau}\left(\frac{19+5\nu_A}{1-\nu_A}\right)\omega^5. \tag{5.97}$$

The sum occurring in the first term on the right-hand side is a finite geometric series, hence we can write (5.97) in the form

$$q = 4\tau\omega\left(\frac{1-\omega^5}{1-\omega}\right) + \frac{1}{2}\left(\frac{5-\nu_A}{1-\nu_A}\right)\omega^3 + \frac{1}{144\tau}\left(\frac{19+5\nu_A}{1-\nu_A}\right)\omega^5. \tag{5.98}$$

Restoring dimensions using the definitions in equations (5.29) and (5.30), noting that $\omega = w_0/a$, we can write this in terms of the physical variables (ignoring the effects of gravity) as

$$p = 4\frac{h}{a}\frac{w_0}{a}\left[\frac{\mathcal{N}}{h}\left(\frac{1-\omega^5}{1-\omega}\right) + \frac{1}{8}\left(\frac{5-\nu_A}{1-\nu_A}\right)\overline{E}\left(\frac{w_0}{a}\right)^2 + \frac{1}{576}\frac{\overline{E}^2 h}{\mathcal{N}}\left(\frac{19+5\nu_A}{1-\nu_A}\right)\left(\frac{w_0}{a}\right)^4\right]. \tag{5.99}$$

Equation (5.99) is similar to Equation (3) of the Beams paper [8] cited in §3.3, if we replace his T_0 by $\mathcal{N}/h$, and replace our finite geometric series by 1. However, the coefficients of his third and fifth-order terms are a factor of 2 greater than ours, and there is a difference in the dependence on Poisson's ratio in the fifth-order term.

Returning now to the general plate perturbation equations (5.60)–(5.63), we assume that the tensile residual membrane stress dominates τ, so that τ is positive. It follows that the parameter γ^2 defined by

$$\gamma^2 \equiv \frac{\tau}{\Delta}, \quad \text{where} \quad \Delta \equiv \delta + \beta_\nu^2 = d - \left(1-\nu_A^2\right)b^2, \tag{5.100}$$

is also positive (the constant Δ was introduced earlier in equation (4.69)). It is convenient, then, to introduce a new independent variable

$$x \equiv \gamma\rho, \tag{5.101}$$

and new perturbation functions defined by

$$W_n(x) \equiv w_n(x/\gamma), \qquad X_n(x) \equiv x_n(x/\gamma). \tag{5.102}$$

Our system of perturbation equations can be written in terms of these new functions, and the variable x, as

$$\omega^1 : \begin{cases} x\,(x\,X_1' + 2\,X_1)' - \beta_\nu\,\gamma^2\,x\left(W_1'' + \frac{W_1'}{x}\right)' = 0, \\[2ex] \beta_\nu\,(xX_1' + 2X_1)' + \delta\,\gamma^2\left(W_1'' + \frac{W_1'}{x}\right)' - \frac{q_1}{2\gamma^2}\,x - W_1'\,\tau = 0, \end{cases} \tag{5.103}$$

$$\omega^2 : \begin{cases} x\,(x\,X_2' + 2\,X_2)' - \beta_\nu\,\gamma^2\,x\left(W_2'' + \frac{W_2'}{x}\right)' + \frac{1}{2}\gamma^2\,(W_1')^2 = 0, \\[2ex] \beta_\nu\,(xX_2' + 2X_2)' + \delta\,\gamma^2\left(W_2'' + \frac{W_2'}{x}\right)' - \frac{q_2}{2\gamma^2}\,x - W_1'\,(X_1 + \tau) - W_2'\,\tau = 0, \end{cases} \tag{5.104}$$

$$\omega^3 : \begin{cases} x\,(x\,X_3' + 2\,X_3)' - \beta_\nu\,\gamma^2\,x\left(W_3'' + \frac{W_3'}{x}\right)' + \gamma^2\,W_1'\,W_2' = 0, \\[2ex] \beta_\nu\,(xX_3' + 2\,X_3)' + \delta\,\gamma^2\left(W_3'' + \frac{W_3'}{x}\right)' - \frac{q_3}{2\gamma^2}\,x \\ \qquad\qquad - W_1'\,(X_2 + \tau) - W_2'\,(X_1 + \tau) - W_3'\,\tau = 0, \end{cases} \tag{5.105}$$

$$\omega^4 : \begin{cases} x\,(x\,X_4' + 2\,X_4)' - \beta_\nu\,\gamma^2\,x\left(W_4'' + \frac{W_4'}{x}\right)' + \frac{1}{2}\gamma^2\left[(W_2')^2 + 2\,W_1'\,W_3'\right] = 0, \\[2ex] \beta_\nu\,(xX_4' + 2\,X_4)' + \delta\,\gamma^2\left(W_4'' + \frac{W_4'}{x}\right)' - \frac{q_4}{2\gamma^2}\,x - W_1'\,(X_3 + \tau) \\ \qquad\qquad - W_2'\,(x_2 + \tau) - W_3'\,(X_1 + \tau) - W_4'\,\tau = 0. \end{cases} \tag{5.106}$$

The boundary conditions for a rigidly clamped edge at $\rho = 1$, i.e, $x = \gamma$, are

$$\gamma\,X_n'(\gamma) + (1 - \nu_A)\,X_n(\gamma) - \gamma^2\,\beta_\nu\,W_n''(\gamma) = 0, \qquad W_n(\gamma) = 0, \quad \text{and} \quad W_n'(\gamma) = 0, \tag{5.107}$$

and we must also satisfy the false boundary conditions on the functions W_n evaluated on axis at $\rho = 0$, equivalent to $x = 0$:

$$W_1(0) = 1, \quad \text{and} \quad W_n(0) = 0, \quad \text{for} \quad n \neq 1. \tag{5.108}$$

From the first equation of (5.103) we have

$$(x\,X_1' + 2\,X_1)' = \beta_\nu\,\gamma^2\left(W_1'' + \frac{W_1'}{x}\right)'. \tag{5.109}$$

This can be substituted in the second equation of (5.103) to obtain

$$\left(W_1'' + \frac{W_1'}{x}\right)' - \frac{q_1}{2\gamma^2\tau}\,x - W_1' = 0,$$

where we used (5.100). This equation can be integrated once, and written as

$$W_1'' + \frac{W_1'}{x} - W_1 = Q_1 x^2 + C_3, \tag{5.110}$$

where C_3 is an arbitrary integration constant, and

$$Q_1 \equiv \frac{1}{4}\frac{q_1}{\gamma^2 \tau}. \tag{5.111}$$

The solution regular at $x = 0$ of the homogeneous differential equation associated with (5.110) is $I_0(x)$, where I_0 is the modified Bessel function of order zero, i.e.,

$$W_{1h}(x) = C_1 I_0(x), \tag{5.112}$$

where C_1 is another integration constant. A particular solution is easily found to be

$$W_{1p}(x) = -Q_1 x^2 + C_2, \tag{5.113}$$

where C_2 is a new arbitrary constant. The general solution is the sum of the homogeneous and particular solutions:

$$W_1(x) = C_1 I_0(x) - Q_1 x^2 + C_2. \tag{5.114}$$

The conditions to be satisfied by W_1 are obtained by setting $n = 1$ in (5.107) and (5.108). Thus,

$$W_1(\gamma) = 0, \quad W_1'(\gamma) = 0, \quad \text{and} \quad W_1(0) = 1. \tag{5.115}$$

The third condition yields $C_2 = 1 - C_1$, and the first condition then gives

$$C_1 = \frac{1 - Q_1\gamma^2}{1 - I_0(\gamma)}. \tag{5.116}$$

Since $C_2 = 1 - C_1$, we thus have

$$C_2 = \frac{Q_1\gamma^2 - I_0(\gamma)}{1 - I_0(\gamma)}. \tag{5.117}$$

The final boundary condition, $W_1'(\gamma) = 0$, will determine Q_1 or, equivalently, q_1. From (5.114) we have

$$W_1'(x) = C_1 I_1(x) - 2Q_1 x, \tag{5.118}$$

noting the identity

$$I_0'(x) \equiv I_1(x). \tag{5.119}$$

Evaluating at $x = \gamma$ yields

$$0 = C_1 I_1(\gamma) - 2Q_1\gamma = \left[\frac{1 - Q_1\gamma^2}{1 - I_0(\gamma)}\right] I_1(\gamma) - 2Q_1\gamma, \tag{5.120}$$

which can be solved for Q_1. Thus,

$$Q_1 = \frac{I_1(\gamma)}{\gamma[\gamma I_1(\gamma) + 2 - 2 I_0(\gamma)]} = \frac{1}{\gamma^2}\left[\frac{10}{9} + \frac{16}{\gamma^2} + \frac{\gamma^2}{1296} + \mathrm{O}(\gamma^4)\right], \tag{5.121}$$

and

$$q_1 = 4\gamma^2 \tau \left[\frac{I_1(\gamma)}{\gamma[\gamma I_1(\gamma) + 2 - 2 I_0(\gamma)]}\right] = 64\,\delta + \frac{40}{9}\tau + \tau\,\mathrm{O}(\gamma^2), \tag{5.122}$$

where the second equality made use of the following power series representations for the Bessel functions:

$$I_0(\gamma) \;=\; 1 + \frac{1}{4}\,\gamma^2 + \frac{1}{64}\,\gamma^4 + \frac{1}{64\cdot 36}\,\gamma^6 + \ldots, \tag{5.123}$$

$$\gamma\, I_1(\gamma) \;=\; \frac{1}{2}\,\gamma^2 + \frac{1}{16}\,\gamma^4 + \frac{1}{64\cdot 6}\,\gamma^6 + \ldots. \tag{5.124}$$

We next integrate (5.109) once to obtain

$$x\,X_1' \,+\, 2\,X_1 \;=\; \beta_\nu\,\gamma^2\,\left(W_1'' \,+\, \frac{W_1'}{x}\right) \,+\, C_4, \tag{5.125}$$

where C_4 is an arbitrary integration constant. From (5.118) we obtain for the second derivative of W_1:

$$W_1''(x) \;=\; C_1\,\left[I_0(x) \,-\, \frac{1}{x}I_1(x)\right] \,-\, 2Q_1 \tag{5.126}$$

using the identitity

$$I_1'(x) \;\equiv\; I_0(x) \,-\, \frac{1}{x}I_1(x). \tag{5.127}$$

Substituting from (5.118) and (5.126) into (5.125), we find

$$x\,X_1' \,+\, 2\,X_1 \;=\; \beta_\nu\,\gamma^2\,C_1\,I_0(x) \,+\, C_5,$$

where C_5 is a new constant. Multiplying thru by x and making use of the identitity

$$xI_0(x) \;\equiv\; [xI_1(x)]', \tag{5.128}$$

we obtain after one integration the general solution for $X_1(x)$:

$$X_1(x) \;=\; \beta_\nu\,\gamma^2\,C_1\,\frac{I_1(x)}{x} \,+\, \frac{C_5}{2} \,+\, \frac{C_6}{x^2}.$$

We set $C_6 = 0$ to insure regularity at $x = 0$, obtaining

$$X_1(x) \;=\; \beta_\nu\,\gamma^2\,C_1\,\frac{I_1(x)}{x} \,+\, \frac{C_5}{2}. \tag{5.129}$$

The arbitrary constant C_5 must be determined by the first boundary condition of (5.107) with $n = 1$, viz.,

$$\gamma\,X_1'(\gamma) \,+\, (1 \,-\, \nu_A)\;X_1(\gamma) \,-\, \gamma^2\,\beta_\nu\,W_1''(\gamma) \;=\; 0. \tag{5.130}$$

From (5.129) we have

$$X_1'(x) \;=\; \beta_\nu\,\gamma^2\,C_1\,\left[\frac{I_0(x)}{x} \,-\, 2\frac{I_1(x)}{x^2}\right], \tag{5.131}$$

and we substitute from (5.131), (5.129), and (5.126), evaluated at $x = \gamma$, into (5.130) to obtain the following expression for C_5:

$$C_5 \;=\; 2\,\left(\frac{\beta_\nu\gamma}{1\,-\,\nu_A}\right)\,[\,\nu_A\,C_1\,I_1(\gamma) \,-\, 2\,\gamma\,Q_1\,]. \tag{5.132}$$

Substitution of this expression into (5.129) completes the solution of the first-order perturbation equations.

In the second-order equations, we have from the first equation of (5.104):

$$(x\,X_2' \,+\, 2\,X_2)' \;=\; \beta_\nu\,\gamma^2\,\left(W_2'' \,+\, \frac{W_2'}{x}\right)' \,-\, \frac{\gamma^2}{2x}\,(W_1')^2, \tag{5.133}$$

which we substitute into the second equation of (5.104), obtaining

$$\left(W_2'' + \frac{W_2'}{x}\right)' - W_2' = 2\,Q_2\,x + W_1' + \frac{W_1'\,X_1}{\tau} + \frac{\beta_\nu\gamma^2}{2\tau x}\,(W_1')^2, \tag{5.134}$$

where

$$Q_2 \equiv \frac{1}{4}\frac{q_2}{\gamma^2\tau}. \tag{5.135}$$

Substituting in the right-hand side of (5.134) for W_1' and X_1 from (5.118) and (5.129), respectively, yields

$$\begin{aligned}\left(W_2'' + \frac{W_2'}{x}\right)' - W_2' = 2\left(Q_1 - Q_2 - \frac{C_5 Q_1}{2\tau} + \frac{\beta_\nu\gamma^2 Q_1^2}{\tau}\right)x \\ + C_1\left[1 + \frac{1}{2\tau}\left(2\beta_\nu\gamma^2 Q_1 - C_5\right)\right] I_1(x) + \frac{3\beta_\nu\gamma^2 C_1^2}{2\tau}\,\frac{I_1^2(x)}{x}.\end{aligned}$$

Using (5.119) and the identity

$$\left[I_0^2(x) - I_1^2(x)\right]' \equiv 2\frac{I_1^2(x)}{x}, \tag{5.136}$$

the last equation can be integrated once to obtain

$$\begin{aligned}W_2'' + \frac{W_2'}{x} - W_2 = C_6 + \left(Q_1 - Q_2 - \frac{C_5 Q_1}{2\tau} + \frac{\beta_\nu\gamma^2 Q_1^2}{\tau}\right)x^2 \\ + C_1\left[1 + \frac{1}{2\tau}\left(2\beta_\nu\gamma^2 Q_1 - C_5\right)\right] I_0(x) + \frac{3\beta_\nu\gamma^2 C_1^2}{4\tau}\left[I_0^2(x) - I_1^2(x)\right],\end{aligned} \tag{5.137}$$

where C_6 is an arbitrary integration constant. The solution of the homogeneous equation associated with (5.137) is again $I_0(x)$. However, we have been unsuccessful in finding an appropriate particular solution, due to the appearance of the quadratic products of Bessel functions in the last term on the right-hand side of (5.137). This brings to a rather abrupt end our attempt to solve the general perturbation equations past the first order. The one useful result is perhaps the first-order expression (5.122) for the dimensionless pressure load:

$$q_1 = 4\,\gamma^2\,\tau\left[\frac{I_1(\gamma)}{\gamma\left[\gamma\,I_1(\gamma) + 2 - 2\,I_0(\gamma)\right]}\right] = 64\,\delta + \frac{40}{9}\tau + \tau\,\mathrm{O}(\gamma^2), \tag{5.138}$$

where the last equality was obtained by using power series representations for the modified Bessel functions. From the definitions of δ and τ we have

$$\delta = \frac{1}{\overline{E}ha^2}\left[D - \left(\frac{AB^2 - 2A_\nu BB_\nu + AB_\nu^2}{A^2 - A_\nu^2}\right)\right], \qquad \tau = \frac{\mathcal{N}}{\overline{E}h}. \tag{5.139}$$

Substituting (5.138) in (5.56) yields the first-order expression for pressure as a function of the dimensionless apex displacement, i.e., $q = q_1\omega$, which can be written using (5.139) and the definitions of q and ω (to restore the dimensions) as

$$\frac{pa}{\overline{E}h} = \left\{\frac{64}{\overline{E}ha^2}\left[D - \left(\frac{AB^2 - 2A_\nu BB_\nu + AB_\nu^2}{A^2 - A_\nu^2}\right)\right] + \frac{40}{9}\frac{\mathcal{N}}{\overline{E}h} + \tau\,\mathrm{O}(\gamma^2)\right\}\frac{w_0}{a}. \tag{5.140}$$

When specialized to a single material in which there is no residual stress (hence $B = 0$, $B_\nu = 0$, and $\mathcal{N} = 0$), equation (5.140) reduces to a form of Chia's first-order result:

$$\frac{pa}{Eh} = \left(\frac{64D}{Eha^2}\right)\frac{w_0}{a}, \tag{5.141}$$

noting that $D = Eh^3/[12(1-\nu^2)]$ for a single material. This also follows directly from the linear plate theory of a single material, as evidenced by evaluating equation (4.189) on-axis at $R = 0$.

It should be noted that one of the major sources of our difficulty involves the appearance of terms containing the coefficient β_ν. We have shown in equation (4.60), and its specialization (4.61) to two layers, that this coefficient can be written as a sum of terms, each of which depends upon a difference of two Poisson's ratios. Thus, if all layers have the same Poisson ratio, this coefficient must vanish, considerably simplifying the resulting equations. It is unlikely that this will ever be exactly true, but it is perhaps worthwhile investigating the perturbation solutions obtained by making the assumption that terms containing β_ν can be set equal to zero. We have found that with such an assumption we gain a solution through second order, but are unable to solve the third-order equations, and so it is here that we end our discussion of the perturbation series approach to finding approximate pressure -vs- axial displacement curves.

5.2.2 Buckling Due to Compressive Intrinsic Stress Loads

For the second type of perturbation problem, we assume the pressure and gravity loads defined by q to be absent, and instead investigate the *compressive* intrinsic coating stress contained in τ required to produce out-of-plane deflections of the laminate from its initially flat state. The perturbation series expansions for $\widehat{w}(\rho)$ and $x_r(\rho)$ are the same as before, i.e., from (5.54) and (5.55):

$$\widehat{w}(\rho) = \sum_{n=1} w_n(\rho)\,\omega^n = w_1(\rho)\,\omega + w_2(\rho)\,\omega^2 + w_3(\rho)\,\omega^3 + \ldots. \tag{5.142}$$

$$x_r(\rho) = \sum_{n=1} x_n(\rho)\,\omega^n = x_1(\rho)\,\omega + x_2(\rho)\,\omega^2 + x_3(\rho)\,\omega^3 + \ldots, \tag{5.143}$$

but we now set $q = 0$ in the equilibrium equations, and construct a perturbation expansion for the intrinsic stress load τ, assumed to be negative, beginning with the $n = 0$ term:

$$\tau = -\sum_{n=0} \tau_n\,\omega^n = -\tau_0 - \tau_1\,\omega - \tau_2\,\omega^2 - \tau_3\,\omega^3 - \ldots. \tag{5.144}$$

The clamped edge boundary conditions on the perturbation functions, as well as the false boundary conditions, are also the same as in (5.66) and (5.67), i.e.,

$$x_n'(1) + (1-\nu_A)\,x_n(1) - \beta_\nu\,w_n''(1) = 0, \qquad w_n(1) = 0, \quad \text{and} \quad w_n'(1) = 0, \tag{5.145}$$

$$w_1(0) = 1, \quad \text{and} \quad w_n(0) = 0, \quad \text{for} \quad n \neq 1. \tag{5.146}$$

The nonlinear term appearing in equation (5.47) is given to fourth order in ω, as earlier, by

$$(\widehat{w}_{,\rho})^2 = (w_1')^2\,\omega^2 + 2\,w_1'\,w_2'\,\omega^3 + \left[(w_2')^2 + 2\,w_1'\,w_3'\right]\,\omega^4 + \ldots. \tag{5.147}$$

but the nonlinear term in (5.53) now involves the perturbation series for τ, and takes the form

$$\begin{aligned}(\widehat{w}_{,\rho})\,(x_r + \tau) = -w_1'\,\tau_0\,\omega + [w_1'\,(x_1-\tau_1) - w_2'\,\tau_0]\,\omega^2 + \\ [w_1'\,(x_2-\tau_2) + w_2'\,(x_1-\tau_1) - w_3'\,\tau_0]\,\omega^3 \\ + [w_1'\,(x_3-\tau_3) + w_2'\,(x_2-t_2) + w_3'\,(x_1-\tau_1) - w_4'\,\tau_0]\,\omega^4 + \ldots.\end{aligned} \tag{5.148}$$

When these perturbation series are substituted in (5.47) and (5.53), we obtain the following equations as coefficients of the indicated powers of ω for the first four orders:

$$\omega^1 : \begin{cases} \rho\,(\rho\,x_1' + 2\,x_1)' - \beta_\nu\,\rho\left(w_1'' + \frac{w_1'}{\rho}\right)' = 0, \\ \beta_\nu\,(\rho x_1' + 2x_1)' + \delta\left(w_1'' + \frac{w_1'}{\rho}\right)' + w_1'\,\tau_0 = 0, \end{cases} \tag{5.149}$$

$$
\omega^2 : \begin{cases} \rho\,(\rho\,x_2' + 2\,x_2)' - \beta_\nu\,\rho\left(w_2'' + \frac{w_2'}{\rho}\right)' + \frac{1}{2}\,(w_1')^2 = 0, \\ \\ \beta_\nu\,(\rho x_2' + 2x_2)' + \delta\left(w_2'' + \frac{w_2'}{\rho}\right)' - w_1'\,(x_1 - \tau_1) + w_2'\,\tau_0 = 0, \end{cases} \tag{5.150}
$$

$$
\omega^3 : \begin{cases} \rho\,(\rho\,x_3' + 2\,x_3)' - \beta_\nu\,\rho\left(w_3'' + \frac{w_3'}{\rho}\right)' + w_1'\,w_2' = 0, \\ \\ \beta_\nu\,(\rho x_3' + 2\,x_3)' + \delta\left(w_3'' + \frac{w_3'}{\rho}\right)' - w_1'\,(x_2 - \tau_2) - w_2'\,(x_1 - \tau_1) + w_3'\,\tau_0 = 0, \end{cases} \tag{5.151}
$$

$$
\omega^4 : \begin{cases} \rho\,(\rho\,x_4' + 2\,x_4)' - \beta_\nu\,\rho\left(w_4'' + \frac{w_4'}{\rho}\right)' + \frac{1}{2}\left[(w_2')^2 + 2\,w_1'\,w_3'\right] = 0, \\ \\ \beta_\nu\,(\rho x_4' + 2\,x_4)' + \delta\left(w_4'' + \frac{w_4'}{\rho}\right)' - w_1'\,(x_3 - \tau_3) \\ \qquad\qquad - w_2'\,(x_2 - \tau_2) - w_3'\,(x_1 - \tau_1) + w_4'\,\tau_0 = 0. \end{cases} \tag{5.152}
$$

Beginning with equations (5.149), we note that the first can be used to eliminate x_1 in the second, yielding the following equation for w_1 after integrating once:

$$
w_1'' + \frac{w_1'}{\rho} + \frac{\tau_0}{\Delta}\,w_1 = C_0, \tag{5.153}
$$

where C_0 is an arbitrary integration constant, and Δ was introduced earlier in equation (4.69), repeated here:

$$
\Delta \equiv \delta + \beta_\nu^2 = d - \left(1 - \nu_A^2\right) b^2. \tag{5.154}
$$

Setting

$$
\lambda^2 \equiv \frac{\tau_0}{\Delta}, \tag{5.155}
$$

equation (5.153) can be written as

$$
w_1'' + \frac{w_1'}{\rho} + \lambda^2\,w_1 = C_0. \tag{5.156}
$$

The solution regular at the origin of the homogeneous equation associated with (5.156) is $J_0(\lambda\rho)$, the ordinary Bessel function of order 0, and the particular solution is simply $C_0/\lambda^2 \equiv C_2$, hence the general solution is

$$
w_1(\rho) = C_1 J_0(\lambda\rho) + C_2\,. \tag{5.157}
$$

This must satisfy the boundary conditions (5.145) and (5.146) with $n = 1$. The conditions $w_1(0) = 1$ and $w_1(1) = 0$ yield the solutions

$$
C_1 = \frac{1}{1 - J_0(\lambda)}, \quad \text{and} \quad C_2 = 1 - C_1. \tag{5.158}
$$

The derivative of (5.157) gives

$$
w_1'(\rho) = -C_1 \lambda J_1(\lambda\rho), \tag{5.159}
$$

and the boundary condition $w_1'(1) = 0$ then requires λ to be a solution of

$$
J_1(\lambda) = 0, \tag{5.160}
$$

that is, λ must be a zero of J_1, the Bessel function of first order. The first (non-zero) zero of J_1 occurs at $\lambda = 3.832$, hence the smallest value of τ_0 is, from (5.155):

$$
\tau_0 = \lambda^2\,\Delta = 14.684\,\Delta. \tag{5.161}
$$

Thus, from (5.161) and (5.144), the critical value of τ above which there will be an out-of-plane displacement, is

$$\tau \;=\; -\tau_0 \;=\; -14.684\,\Delta. \tag{5.162}$$

Restoring the dimensional constants using the definitions in §5.1, equation (5.162) yields the following value for the critical intrinsic stress resultant:

$$\mathcal{N}_0 \;=\; -\frac{14.684}{a^2}\left(D \,-\, \frac{B^2}{A}\right). \tag{5.163}$$

For a single-layer coating, we have $\mathcal{N}_0 = h_c S_{c0} + h_s S_s$, hence the critical coating stress S_{c0} required for an out-of-plane displacement is

$$S_{c0} \;=\; -\frac{h_s}{h_c}\,S_s \,-\, \frac{14.684}{h_c a^2}\left(D \,-\, \frac{B^2}{A}\right). \tag{5.164}$$

Although we have succeeded in continuing the solutions of the perturbation series to higher orders, it is this result for the critical coating stress that is of greatest interest to us. We thus end our discussion of perturbation methods here, and proceed to a brief discussion of solutions in the form of power series.

5.3 Power Series Solutions

In 1934, Way [34] published "exact" power series solutions for the geometrically nonlinear plate. In order to follow his work as closely as possible, we introduce his normalizations of the radial variable and displacement components with respect to the thickness h rather than the radius a, denoting them by

$$\xi \equiv \frac{R}{h} = \frac{a}{h}\,\rho, \qquad U \equiv \frac{u}{h} = \frac{a}{h}\,\widehat{u}, \qquad W \equiv \frac{w}{h} = \frac{a}{h}\,\widehat{w}. \tag{5.165}$$

Note the correspondence with Way's notation: our ξ is his u-coordinate and our radial displacement u is his variable ρ. Our variables x_r and x_θ correspond directly (in the absence of intrinsic stress loads) to Way's variables S_r' and S_t', respectively, which we distinguish here by X_r and X_θ, respectively, so that

$$X_r \;\equiv\; x_r \;=\; \frac{N_R - \mathcal{N}}{\overline{E}\,h}, \qquad X_\theta \;\equiv\; x_\theta \;=\; \frac{N_\Theta - \mathcal{N}}{\overline{E}\,h}. \tag{5.166}$$

We also introduce new dimensionless curvatures K_r and K_θ defined by

$$K_r \;\equiv\; h\,k_{RR} \;=\; W_{,\xi\xi} \;=\; \frac{h}{a}\,\kappa_r \qquad K_\theta \;\equiv\; h\,k_{\Theta\Theta} \;=\; \frac{W_{,\xi}}{\xi} \;=\; \frac{h}{a}\,\kappa_\theta. \tag{5.167}$$

Way did not consider the effects of gravity. His dimensionless pressure load, which we denote by Q (not to be confused with Way's scale-invariant pressure load Q) is related to our q (in the absence of gravitational effects) by

$$Q \equiv \frac{p}{\overline{E}} = \frac{h}{a}\,q. \tag{5.168}$$

The dimensionless stress load τ does not appear in his equations, since he is not concerned with intrinsic stresses.

The equations we must solve are the compatibility equation (5.47) and axial equilibrium equation (5.53), viz.,

$$\rho\,[x_r \,+\, x_\theta \,-\, \beta_\nu\,(\kappa_r \,+\, \kappa_\theta)]_{,\rho} \;=\; -\frac{1}{2}\widehat{w}_{,\rho}^2, \tag{5.169}$$

and

$$-\beta_\nu\,(x_r \,+\, x_\theta)_{,\rho} \,-\, \delta\,(\kappa_r \,+\, \kappa_\theta)_{,\rho} \,+\, \widehat{w}_{,\rho}\,(\tau \,+\, x_r) + q\,\frac{\rho}{2} \;=\; 0, \tag{5.170}$$

using (5.37) to eliminate x_θ, i.e.,

$$x_\theta = (\rho\, x_r)_{,\rho}. \tag{5.171}$$

In terms of Way's variables, these take the forms

$$\xi\,[X_r + X_\theta - \mathcal{B}_\nu\,(K_r + K_\theta)]_{,\xi} = -\frac{1}{2}W_{,\xi}^2. \tag{5.172}$$

and

$$-\mathcal{B}_\nu\,(X_r + X_\theta)_{,\xi} - \mathcal{D}\,(K_r + K_\theta)_{,\xi} + W_{,\xi}\,(\tau + X_r) + Q\,\frac{\xi}{2} = 0, \tag{5.173}$$

where

$$X_\theta = (\xi\, X_r)_{,\xi}, \equiv X_r + \xi\, X_{r,\xi} \tag{5.174}$$

and we have introduced new constants

$$\mathcal{B}_\nu \equiv \frac{a}{h}\,\beta_\nu = \frac{1}{\overline{E}\,h^2}\,(B_\nu - \nu_A\,B), \qquad \mathcal{D} \equiv \left(\frac{a}{h}\right)^2\delta = \frac{1}{\overline{E}\,h^3}\left[D - \frac{1}{\overline{E}\,h}\,(B^2 - \nu_A\,B\,B_\nu + B_\nu^2)\right], \tag{5.175}$$

using definitions given earlier. Note that the new constants do not depend on the radius a, and that the terms proportional to $\mathcal{B}_\nu$ do not appear in Way's equations, as B_ν and B are zero for a plate consisting of a *single* material. Also, for a single material D reduces to the usual bending stiffness $Eh^3/[12(1-\nu^2)]$, hence $\mathcal{D}$ reduces to $1/[12(1-\nu^2)]$. It is easy to show that the radial displacement (5.46) in Way's variables takes the form

$$U = \xi\,(X_\theta - \nu_A\,X_r - \mathcal{B}_\nu\,K_r - \mathcal{B}\,K_\theta), \tag{5.176}$$

where the constant $\mathcal{B}$ is defined by

$$\mathcal{B} \equiv \frac{a}{h}\,\beta. \tag{5.177}$$

Following Way [34], we seek power series solutions for X_r and $W_{,\xi}$ having the forms

$$X_r(\xi) = -\tau + \sum_{n=0}^{\infty} b_{2n}\,\xi^{2n}, \tag{5.178}$$

$$W_{,\xi}(\xi) = \sqrt{8}\,\sum_{n=0}^{\infty} c_{2n+1}\,\xi^{2n+1}. \tag{5.179}$$

The series for W is found by integrating (5.179) term-by-term to yield

$$W(\xi) = c_0 + \sqrt{8}\,\sum_{n=0}^{\infty}\left(\frac{1}{2n+2}\right) c_{2n+1}\,\xi^{2n+2}, \tag{5.180}$$

where c_0 is an arbitrary integration constant (the value of W when $\xi = 0$). To determine the coefficients b_{2n} and c_{2n+1}, $n = 0, 1, 2\ldots$, we note that from (5.174) and (5.178) we have

$$X_\theta = (\xi X_r)_{,\xi} = \left(-\xi\,\tau + \sum_{n=0}^{\infty} b_{2n}\,\xi^{2n+1}\right)_{,\xi} = -\tau + \sum_{n=0}^{\infty}(2n+1)\,b_{2n}\,\xi^{2n}, \tag{5.181}$$

hence

$$(X_r + X_\theta)_{,\xi} = \left[-2\tau + 2\sum_{n=0}^{\infty}(n+1)\,b_{2n}\,\xi^{2n}\right]_{,\xi} = 4\sum_{n=0}^{\infty} n(n+1)\,b_{2n}\,\xi^{2n-1}. \tag{5.182}$$

From (5.179) we obtain

$$W_{,\xi\xi} = \sqrt{8}\,\sum_{n=0}^{\infty}(2n+1)\,c_{2n+1}\,\xi^{2n}, \tag{5.183}$$

hence from the definitions (5.167):

$$K_r = W_{,\xi\xi} = \sqrt{8} \sum_{n=0}^{\infty} (2n+1)\, c_{2n+1}\, \xi^{2n}, \qquad K_\theta = \frac{W_{,\xi}}{\xi} = \sqrt{8} \sum_{n=0}^{\infty} c_{2n+1}\, \xi^{2n}. \tag{5.184}$$

From these two expressions it follows that

$$(K_r + K_\theta)_{,\xi} = 2\sqrt{8} \left[\sum_{n=0}^{\infty} (n+1)\, c_{2n+1}\, \xi^{2n} \right]_{,\xi} = 4\sqrt{8} \sum_{n=0}^{\infty} n(n+1)\, c_{2n+1}\, \xi^{2n-1}. \tag{5.185}$$

Substituting the appropriate series expressions in the compatibility condition (5.172) and axial equilibrium equation (5.173), we obtain

$$\sum_{n=0}^{\infty} n(n+1) \left(b_{2n} - \sqrt{8}\, \mathcal{B}_\nu\, c_{2n+1} \right) \xi^{2n} + \left(\sum_{n=0}^{\infty} c_{2n+1}\, \xi^{2n+1} \right)^2 = 0, \tag{5.186}$$

and

$$-8 \sum_{n=0}^{\infty} n(n+1) \left(\mathcal{B}_\nu\, b_{2n} + \sqrt{8}\, \mathcal{D}\, c_{2n+1} \right) \xi^{2n} + 2\sqrt{8} \sum_{n=0}^{\infty} c_{2n+1}\, \xi^{2n+1} \cdot \sum_{n=0}^{\infty} b_{2n}\, \xi^{2n} + Q\, \xi = 0, \tag{5.187}$$

respectively. Using *Mathematica* again, we determined the following recursion relations for all coefficients in terms of the two unknown coefficients b_0 and c_1:

$$b_{2k+2} = \frac{\mathcal{D}}{(\mathcal{D} + \mathcal{B}_\nu^2)} B_{2k+2} + \frac{\sqrt{8}\, \mathcal{B}_\nu}{(\mathcal{D} + \mathcal{B}_\nu^2)} C_{2k+3}, \quad k = 0, 1, 2, \ldots \tag{5.188}$$

$$c_{2k+3} = -\frac{\sqrt{8}\, \mathcal{B}_\nu}{8(\mathcal{D} + \mathcal{B}_\nu^2)} B_{2k+2} + \frac{1}{(\mathcal{D} + \mathcal{B}_\nu^2)} C_{2k+3}, \quad k = 0, 1, 2, \ldots \tag{5.189}$$

where

$$C_3 = \frac{1}{8} \left(\frac{Q}{2\sqrt{8}} + b_0\, c_1 \right), \quad C_{2k+1} = \frac{1}{[(2k+1)^2 - 1]} \sum_{m=0}^{k-1} B_{2m}\, C_{2k-1-2m}, \quad k = 2, 3, 4, \ldots \tag{5.190}$$

and

$$B_{2k+2} = -\frac{1}{(k+1)(k+2)} \sum_{m=0}^{k} C_{2m+1}\, C_{2k+1-2m}, \quad k = 0, 1, 2, \ldots \tag{5.191}$$

are essentially Way's [34] coefficients, with his factor $12(1-\nu^2)$ omitted in the numerators of the C-coefficients (as mentioned earlier, this factor corresponds roughly to $1/\mathcal{D}$ in our coefficients). Our unknown coefficients correspond directly to his B_0 and C_1, i.e., $b_0 = B_0$ and $c_1 = C_1$. Thus, for example, when $k = 0$ in (5.188) and (5.189) we obtain

$$b_2 = -\frac{\mathcal{D}}{(\mathcal{D} + \mathcal{B}_\nu^2)} \frac{1}{2} c_1^2 + \frac{\sqrt{8}\, \mathcal{B}_\nu}{(\mathcal{D} + \mathcal{B}_\nu^2)} \frac{1}{8} \left(\frac{Q}{2\sqrt{8}} + b_0\, c_1 \right), \tag{5.192}$$

$$c_3 = \frac{\sqrt{8}\, \mathcal{B}_\nu}{8(\mathcal{D} + \mathcal{B}_\nu^2)} \frac{1}{2} c_1^2 + \frac{1}{(\mathcal{D} + \mathcal{B}_\nu^2)} \frac{1}{8} \left(\frac{Q}{2\sqrt{8}} + b_0\, c_1 \right), \tag{5.193}$$

where we used (5.191) with $k = 0$, i.e., $B_2 = -c_1^2/2$, noting that $c_1^2 = C_1^2$.

The coefficients b_0 and c_1 must be determined, in general, from the boundary conditions. We consider here a clamped boundary, for which the boundary conditions at the circular edge $R = a$, corresponding to $\xi = a/h \equiv \xi_0$ (Way's constant u_0), are

$$U(\xi_0) = 0, \quad W(\xi_0) = 0, \quad \text{and} \quad W_{,\xi}(\xi_0) = 0. \tag{5.194}$$

From equation (5.176), we have

$$U(\xi_0) = \xi_0 \left[X_\theta(\xi_0) - \nu_A X_r(\xi_0) - \mathcal{B}_\nu K_r(\xi_0) \right], \tag{5.195}$$

noting that K_θ is proportional to $W_{,\xi}$, which must vanish at $\xi = \xi_0$ according to the second boundary condition of (5.194). Thus, the first boundary condition corresponds to

$$X_\theta(\xi_0) - \nu_A X_r(\xi_0) - \mathcal{B}_\nu K_r(\xi_0) = 0. \tag{5.196}$$

Substituting in (5.196) the appropriate series expansions from (5.181), (5.178), and (5.184), evaluated at $\xi = \xi_0$, yields for the first boundary condition:

$$\sum_{n=0}^{\infty} \Big[(2n + 1 - \nu_A) b_{2n} - \sqrt{8}\, \mathcal{B}_\nu (2n+1) c_{2n+1} \Big] \xi_0^{2n} - (1 - \nu_A)\, \tau = 0. \tag{5.197}$$

From (5.179) evaluated at $\xi = \xi_0$, the third boundary condition of (5.194) takes the simple form

$$\sum_{n=0}^{\infty} c_{2n+1} \xi_0^{2n+1} = 0. \tag{5.198}$$

The two nonlinear equations (5.197) and (5.198) must be solved simultaneously for the unknown coefficients b_0 and c_1, and then the second boundary condition of (5.194) yields the remaining unknown coefficient c_0, i.e.,

$$c_0 = -\sqrt{8} \sum_{n=0}^{\infty} \left(\frac{1}{2n+2} \right) c_{2n+1} \xi_0^{2n+2}, \tag{5.199}$$

according to (5.180). We note that the coefficients b_0 and c_1 are, according to (5.178) and (5.183), related to the values of X_r and $W_{,\xi\xi}$ on-axis at $\xi = 0$ by

$$b_0 = X_r(0) + \tau, \quad \text{and} \quad c_1 = \frac{\sqrt{8}}{8} W_{,\xi\xi}(0), \tag{5.200}$$

respectively. Since $W_{,\xi\xi}$ is proportional to the curvature of the axial displacement, we expect that if the pressure load changes sign (reversing the curvature), so also must c_1.

The solution of the system of two nonlinear equations (5.197) and (5.198) for b_0 and c_1 is a daunting task, unless one is able to provide good initial estimates for them. Way partially avoided this difficulty by first assigning values to b_0 and c_1, then solving (5.198) for the corresponding value of $\xi_0 = a/h$ (again, his u_0). In this way, the second boundary condition was automatically satisfied. This value of ξ_0 must, however, produce a radius a, for a given specimen thickness h, *near* the actual radii of test articles in any experiment to which the theory is applied. The burden of solving (5.197) and (5.198) directly is thus shifted to that of choosing values for b_0 and c_1 that lead to a value of ξ_0 which provides a suitable a for a given h. The solution of this difficulty was unfortunately not elaborated upon in Reference [34]. We have found, however, that for values of ξ_0 of interest to Way, i.e., for $\xi_0 \lesssim 100$, approximate values for b_0 and c_1 can be obtained by keeping only the first two terms of the boundary conditions (5.197) and (5.198), and solving the truncated equations for b_0 and c_1 (this procedure leads to a cubic equation for c_1). At any rate, for a given b_0, Way proceeded to determine two or three more values of c_1 that would produce values of ξ_0 yielding radii in an interval containing the desired radius a. Finally, he used these values to determine *by interpolation* the value of c_1 that would give a ξ_0 satisfying the first boundary condition (5.197), and fixing the radius a. All other quantities could be determined in the form of power series, using the values for b_0 and c_1 satisfying the required boundary conditions.

Unfortunately, values of ξ_0 that correspond to typical coated membrane geometries are *much higher* than any considered by Way. For example, the membrane model discussed in §3.3 has a radius of $a = 0.0762$ m and total thickness $h = 21\,\mu$m, corresponding to $\xi_0 \approx 3629$, nearly two orders of magnitude higher than the values of interest to Way. For large aperture coated membrane reflectors having radii on the order of a meter or more, ξ_0 is another one to two orders of magnitude greater still! The method of truncating each of the two boundary conditions at the second term and solving for b_0 and c_1 yields completely erroneous estimates of these coefficients for such large values of ξ_0. We have been thus far unsuccessful in developing a method for estimating b_0 and c_1 that is applicable to coated membrane geometries.

5.3.1 Scale-Invariant Functions and Constants

Consider a transformation of the dimensionless coordinate ξ by an arbitrary scale factor s, generating a scaled coordinate $\overline{\xi}$:

$$\overline{\xi} = s\xi. \tag{5.201}$$

We assume that the functions and constants appearing in equations (5.172)–(5.174) transform to new functions under this scaling according to the following general rules:

$$\overline{W}(\overline{\xi}) = s^p\, W(\xi), \quad \overline{K}_r(\overline{\xi}) = s^{\ell_1}\, K_r(\xi), \quad \overline{K}_\theta(\overline{\xi}) = s^{\ell_2}\, K_\theta(\xi), \tag{5.202}$$

$$\overline{X}_r(\overline{\xi}) = s^{k_1}\, X_r(\xi), \quad \overline{X}_\theta(\overline{\xi}) = s^{k_2}\, X_\theta(\xi), \tag{5.203}$$

$$\overline{\mathcal{B}}_\nu = s^m\, \mathcal{B}_\nu, \quad \overline{\mathcal{D}} = s^n\, \mathcal{D}, \quad \overline{\tau} = s^r\, \tau, \quad \overline{Q} = s^t\, Q, \tag{5.204}$$

where the scale factor exponents are arbitrary. For any scaled function $\overline{f}(\overline{\xi}) = s^\alpha\, f(\xi)$, we find using the chain rule for differentiation the following relations between derivatives with respect to the scaled and unscaled coordinates:

$$\overline{f}_{,\overline{\xi}} = s^{\alpha-1}\, f_{,\xi} \Leftrightarrow f_{,\xi} = s^{1-\alpha}\, \overline{f}_{,\overline{\xi}}. \tag{5.205}$$

Under these scalings, then, the fundamental equations (5.172)-(5.174) transform to

$$s^{-1}\overline{\xi}\,\left[\left(s^{1-k_1}\overline{X}_r + s^{1-k_2}\overline{X}_\theta\right) - s^{-m}\overline{\mathcal{B}}_\nu\,\left(s^{1-\ell_1}\overline{K}_r + s^{1-\ell_2}\overline{K}_\theta\right)\right]_{,\overline{\xi}} = -\frac{1}{2}s^{2-2p}\,\overline{W}^2_{,\overline{\xi}}, \tag{5.206}$$

$$\begin{aligned} - s^{-m}\overline{\mathcal{B}}_\nu\,\left(s^{1-k_1}\overline{X}_r + s^{1-k_2}\overline{X}_\theta\right)_{,\overline{\xi}} - s^{-n}\overline{\mathcal{D}}\,\left(s^{1-\ell_1}\overline{K}_r + s^{1-\ell_2}\overline{K}_\theta\right)_{,\overline{\xi}} \\ + s^{1-p}W_{,\overline{\xi}}\left(s^{-r}\overline{\tau} + s^{-k_1}\overline{X}_r\right) + s^{-1-t}\overline{Q}\,\frac{\overline{\xi}}{2} = 0, \end{aligned} \tag{5.207}$$

and

$$s^{k_2}\overline{X}_\theta = s^{k_1}\overline{X}_r + s^{k_1}\overline{\xi}\,\overline{X}_{r,\overline{\xi}}. \tag{5.208}$$

These equations can be simplified to

$$\overline{\xi}\,\left[\left(\overline{X}_r + s^{k_1-k_2}\overline{X}_\theta\right) - \overline{\mathcal{B}}_\nu\,\left(s^{k_1-m-\ell_1}\,\overline{K}_r + s^{k_1-m-\ell_2}\,\overline{K}_\theta\right)\right]_{,\overline{\xi}} = -\frac{1}{2}s^{k_1+2-2p}\,\overline{W}^2_{,\overline{\xi}}, \tag{5.209}$$

$$\begin{aligned} \overline{\mathcal{B}}_\nu\,\left(\overline{X}_r + s^{k_1-k_2}\overline{X}_\theta\right)_{,\overline{\xi}} - \overline{\mathcal{D}}\,\left(s^{k_1+m-n-\ell_1}\,\overline{K}_r + s^{k_1+m-n-\ell_2}\,\overline{K}_\theta\right)_{,\overline{\xi}} \\ + W_{,\overline{\xi}}\left(s^{k_1+m-p-r}\overline{\tau} + s^{m-p}\overline{X}_r\right) + s^{k_1+m-2-t}\,\overline{Q}\,\frac{\overline{\xi}}{2} = 0, \end{aligned} \tag{5.210}$$

and

$$\overline{X}_\theta = s^{k_1-k_2}\left(\overline{\xi}\,\overline{X}_r\right)_{,\overline{\xi}}, \tag{5.211}$$

respectively. These equations will retain the same form in the scaled variables as in the unscaled variables if, and only if, the exponent of each occurrence of the scale factor s is zero. The equations are then said to be *form-invariant* under the scaling transformation. This requirement leads to a linear system of equations which can be solved for eight of the nine exponents in terms of the remaining one. For example, we find the following solutions in terms of m: $k_1 = k_2 = r = 2m-2$; $\ell_1 = \ell_2 = m-2$; $n = 2m$; $p = m$; and $t = 3m-4$. Thus, the fundamental equations are form-invariant under the following one-parameter group (parameter m) of scaling transformations:

$$\overline{W}(\overline{\xi}) = s^m\,W(\xi), \quad \overline{K}_r(\overline{\xi}) = s^{m-2}\,K_r(\xi), \quad \overline{K}_\theta(\overline{\xi}) = s^{m-2}\,K_\theta(\xi), \tag{5.212}$$

$$\overline{X}_r(\overline{\xi}) = s^{2m-2}\,X_r(\xi), \quad \overline{X}_\theta(\overline{\xi}) = s^{2m-2}\,X_\theta(\xi), \tag{5.213}$$

$$\overline{\tau} = s^{2m-2}\,\tau, \quad \overline{\mathcal{B}}_\nu = s^m\,\mathcal{B}_\nu, \quad \overline{\mathcal{D}} = s^{2m}\,\mathcal{D}, \quad \overline{Q} = s^{3m-4}\,Q. \tag{5.214}$$

The subgroup of this transformation that preserves the material constants $\mathcal{B}_\nu$ and $\mathcal{D}$, i.e., leaves them scale-invariant, is obtained by setting $m = 0$. In this case the dimensionless axial displacement $W(\xi)$ is scale-invariant as well, i.e.,

$$\overline{W}(\overline{\xi}) = W(\xi), \quad \overline{\mathcal{B}}_\nu = \mathcal{B}_\nu, \quad \overline{\mathcal{D}} = \mathcal{D}. \tag{5.215}$$

The remaining functions and constants transform under this subgroup to

$$\overline{K}_r(\overline{\xi}) = s^{-2}\,K_r(\xi), \quad \overline{K}_\theta(\overline{\xi}) = s^{-2}\,K_\theta(\xi), \tag{5.216}$$

$$\overline{X}_r(\overline{\xi}) = s^{-2}\,X_r(\xi), \quad \overline{X}_\theta(\overline{\xi}) = s^{-2}\,X_\theta(\xi), \tag{5.217}$$

$$\overline{\tau} = s^{-2}\,\tau, \quad \overline{Q} = s^{-4}\,Q. \tag{5.218}$$

Using (5.201) to eliminate the scale factor $s = \overline{\xi}/\xi$ in this system, we find the following scale-invariant combinations:

$$\overline{\xi}^2\,\overline{K}_r(\overline{\xi}) = \xi^2\,K_r(\xi), \quad \overline{\xi}^2\,\overline{K}_\theta(\overline{\xi}) = \xi^2\,K_\theta(\xi), \tag{5.219}$$

$$\overline{\xi}^2\,\overline{X}_r(\overline{\xi}) = \xi^2\,X_r(\xi), \quad \overline{\xi}^2\,\overline{X}_\theta(\overline{\xi}) = \xi^2\,X_\theta(\xi), \tag{5.220}$$

$$\overline{\xi}^2\,\overline{\tau} = \xi^2\,\tau, \quad \overline{\xi}^4\,\overline{Q} = \xi^4\,Q. \tag{5.221}$$

These expressions apply at all points including the edge $\xi_0 = a/h$ hence, in particular, $Q\xi_0^4$ is a scale-invariant constant (under the transformations that leave $\mathcal{B}_\nu$ and $\mathcal{D}$ scale-invariant), and the stress-related functions $\xi^2\,X_r$ and $\xi^2\,X_\theta$ are also scale-invariant. Way [34] used $Q\xi_0^4$ (his qu_0^4) as the abscissa in several graphs, and discussed scaling properties briefly in the Appendix of [34].

6 Conclusions

This work is the sequel to a Report [1] in which asymptotic methods were used to derive theories that would aid in understanding the mechanical behavior of a stress-coated membrane. We have applied those theories to a number of boundary value problems, obtaining generalizations of well-known solutions for a membrane, plate, or shell of a single material to solutions for the same structure, but now consisting of a multilayer-coated polymer material. In particular, we (i) found geometrically linear non-axisymmetric solutions for a pressurized membrane laminate having a general Θ-dependent boundary curve, (ii) determined linear membrane vibration solutions for a membrane laminate clamped along a planar, circular boundary, (iii) generalized the Hencky-Campbell [6, 7] geometrically nonlinear power series solutions for a pressurized membrane to a membrane laminate, and discussed the application of these solutions to the interpretation of bulge test data, (iv) derived geometrically linear general solutions for both initially parabolic and initially flat coated membrane laminates, generalizing the Stoney formula for such a structure simply supported at its center, and (v) initiated a search for perturbation series and power series solutions of the geometrically nonlinear theory of an initially flat coated membrane laminate.

Perhaps the most significant accomplishment of this work was the discovery of simple prescriptions for the coating stress that would maintain the shape of an initially parabolic coated membrane after removal from the mold upon which it was cast. The details of the solutions, from which these prescriptions follow, are presented in §4.2 and in particular in §4.2.4. These solutions involve linear combinations of Kelvin functions. Coating prescriptions are given there for membrane laminates both with, and without, pressure and gravitational loads. The prescriptions are presently being used in the preliminary design of a near net-shape stress-coated membrane, and it is hoped that such a structure will be demonstrated for the first time in the very near future.

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A Elementary Analysis of Stress and Strain Due to CTE Mismatch Between Membrane and Mandrel

We consider a polymeric membrane cast on a mandrel, and allowed to cure. The membrane/mandrel laminate is then subjected to an annealing process in which it is raised to a temperature near the glass transition temperature T_g of the membrane. To analyze the subsequent development of stress and strain in the two materials, we use the following pair of simple constitutive relations (which follow, for an axisymmetric system, as the leading order constitutive relations of each theory developed in [1]):

$$S_{RRi} = S_i + \frac{E_i}{1-\nu_i^2}\left(\epsilon_{RR} + \nu_i \epsilon_{\Theta\Theta}\right), \tag{A.1}$$

$$S_{\Theta\Theta i} = S_i + \frac{E_i}{1-\nu_i^2}\left(\epsilon_{\Theta\Theta} + \nu_i \epsilon_{RR}\right), \tag{A.2}$$

where S_i is the in-plane *residual* stress in material i, given by

$$S_i = S_i^I - \frac{E_i}{1-\nu_i}\alpha_i\,(T - T_0). \tag{A.3}$$

Here, we have assumed that the residual stress can manifest either as an *intrinsic* stress S_i^I, associated typically with microstructural changes occurring in a material, or as a *thermal* stress that arises if the temperature T is different from the temperature T_0 at which the stress and strain of the material is purely mechanical. For thermal stress, α_i is the coefficient of thermal expansion (CTE) of material i.

We take the membrane/mandrel laminate configuration at the elevated temperature T_g as the strain-free reference configuration, and compute subsequent deformations from this state. We assume there to be no *intrinsic* stresses in either of these materials, so that after cooling to ambient temperature T the residual stresses in each are purely thermal, given by

$$S_s^o = -\frac{E_s}{1-\nu_s}\alpha_s\,(T-T_g), \quad \text{and} \quad S_m^o = -\frac{E_m}{1-\nu_m}\alpha_m\,(T-T_g). \tag{A.4}$$

Here, the s-subscript refers to the membrane (since it will be the *substrate* for an optical coating), and the m-subscript refers to the *mandrel*. The o-superscript distinguishes variables *prior to coating and releasing* the coated membrane from the mandrel.

The mandrel is assumed to undergo a free contraction during cool-down to a *stress-free* state at the ambient temperature T, so that for $i = m$ the left-hand sides of equations (A.1) and (A.2) are zero, i.e., $S^o_{RRm} = S^o_{\Theta\Theta m} = 0$, hence

$$0 = S_m^o + \frac{E_m}{1-\nu_m^2}\,(\epsilon^o_{RR} + \nu_m\epsilon^o_{\Theta\Theta}), \tag{A.5}$$

$$0 = S_m^o + \frac{E_m}{1-\nu_m^2}\,(\epsilon^o_{\Theta\Theta} + \nu_m\epsilon^o_{RR}). \tag{A.6}$$

These equations are easily solved for the two strain components, yielding

$$\epsilon^o_{RR} = \epsilon^o_{\Theta\Theta} = -S_m^o\left(\frac{1-\nu_m}{E_m}\right) = \alpha_m\,(T-T_g), \tag{A.7}$$

where the final result made use of the second equation of (A.4). Substituting these results for the strain components into (A.1) and (A.2), with $i = s$, then yields the stress components in the membrane substrate, i.e.,

$$\begin{aligned} S^o_{RRs} = S^o_{\Theta\Theta s} &= S_s^o + \frac{E_s}{1-\nu_s^2}\,(1+\nu_s)\,\alpha_m\,(T-T_g) \\ &= -\frac{E_s}{1-\nu_s}\,\alpha_s\,(T-T_g) + \frac{E_s}{1-\nu_s}\,\alpha_m\,(T-T_g) \\ &= -\frac{E_s}{1-\nu_s}\,(\alpha_s-\alpha_m)\,(T-T_g), \end{aligned} \tag{A.8}$$

where the first equation of (A.4) was used to get the second line. For convenience, we write the last equation as

$$S_s = -\frac{E_s}{1-\nu_s}\,\epsilon_s, \tag{A.9}$$

where

$$S_s \equiv S^o_{RRs} = S^o_{\Theta\Theta s}, \tag{A.10}$$

and

$$\epsilon_s \equiv (\alpha_s-\alpha_m)\,(T-T_g) \tag{A.11}$$

is the *recoverable* strain, i.e., the strain the membrane would experience if *released* from the mold and allowed to come to a stress-free state.

As an example, we consider a CP1-DE membrane material (manufactured by SRS Technologies, Inc, Huntsville, AL) cast on an aluminum mandrel. SRS has provided the following material properties for CP1-DE:

$$E_s = 2.172\ \text{GPa}, \tag{A.12}$$

$$\nu_s = 0.34, \tag{A.13}$$

$$\alpha_s = 51.2\ \text{ppm}/\,^\circ\text{C}, \tag{A.14}$$

$$\rho_s = 1.434\ \text{g/cc}, \tag{A.15}$$

$$T_g = 263\ ^\circ\text{C}. \tag{A.16}$$

From the MatWeb website, we find that aluminum alloys have CTE's of around 24 ppm/ °C. Assuming the final temperature to be an ambient room temperature of $T = 20$ °C, the recoverable membrane strain from (A.11) is

$$\epsilon_s = (\alpha_s - \alpha_m)\,\Delta T = 27.2 \times 10^{-6}/°\mathrm{C} \times (-243\,°\mathrm{C}) = -0.00661, \tag{A.17}$$

or roughly 0.7% strain. From (A.17) and (A.9) the residual thermal stress in the membrane is then

$$S_s = -\frac{E_s}{1-\nu_s}\,\epsilon_s = 3.291 \times 10^9 \times 0.00661 = 21.754 \times 10^6 \text{ Pa} = 21.754 \text{ MPa}, \tag{A.18}$$

which we note is positive, hence a tensile stress.

Suppose, now, that a single optical coating is applied to the membrane, and assumed to be perfectly bonded to the membrane (which itself remains bonded to the mandrel). For generality, we assume the coating is placed on the membrane at a temperature T_c, either above or below the temperature T at which the mandrel is stress-free. When the coated membrane/mandrel is returned to the temperature T, the strain components are given by (A.7), assuming the mandrel to again be stress-free, but with T_g replaced by T_c. Substituting these strain components in (A.1) and (A.2), now with $i = c$, yields the stress components in the coating at temperature T:

$$S^o_{RRc} = S^o_{\Theta\Theta c} = S^I_c - \frac{E_c}{1-\nu_c}\,(\alpha_c - \alpha_m)\,(T - T_c)\,, \tag{A.19}$$

where S^I_c is the intrinsic stress in the coating.

When the coated membrane is released from the mandrel, equations (A.1) and (A.2) again apply, but with the *residual* stresses given by (A.8) for the membrane, and by (A.19) for the coating, i.e.,

$$S_s = S^o_{RRs} = S^o_{\Theta\Theta s} = -\frac{E_s}{1-\nu_s}\,(\alpha_s - \alpha_m)\,(T - T_g)\,, \tag{A.20}$$

$$S_c = S^o_{RRc} = S^o_{\Theta\Theta c} = S^I_c - \frac{E_c}{1-\nu_c}\,(\alpha_c - \alpha_m)\,(T - T_c)\,. \tag{A.21}$$

If the coated membrane is first attached to a rigid boundary ring, and then released from the mandrel, the boundary conditions it must satisfy are those of a rigidly clamped edge. From the geometrically linear theory of a stress-coated net-shape film with a clamped boundary, the "on-design" *residual* coating stress maintaining the initial parabolic shape of the coated membrane after releasing it from the mandrel is derivable from equation (4.171), i.e.,

$$\mathcal{N} - 2aF^{\#}\,(p + \gamma_0\,g) = 0, \tag{A.22}$$

where we have used (4.64) and (1.16) to replace κ_0 in terms of the f-number. For a two-layer system with no pressure difference, and ignoring the effects of gravity, this reduces to

$$\mathcal{N} \equiv h_c S_c + h_s S_s = 0, \tag{A.23}$$

where h_s is the membrane thickness and h_c is the coating thickness, hence the required coating stress is

$$S_c = -\frac{h_s}{h_c}\,S_s. \tag{A.24}$$

Assuming for the present that the membrane was coated at the temperature $T_c = T$ where the mandrel was stress-free, the thermal term in (A.21) is zero, hence the residual coating stress is purely intrinsic:

$$S_c = S^I_c = -\frac{h_s}{h_c}\,S_s. \tag{A.25}$$

Thus, for a 1 μm thick coating on a 10 μm thick membrane, and a CTE mismatch stress in the membrane given by (A.18), the required coating stress would be

$$S^I_c = -\frac{10}{1} \times 21.754 \text{ MPa} = -218 \text{ MPa}. \tag{A.26}$$

Suppose, however, that this prescription stress level S_c^I is not precisely met, but acquires some value $S_c = (1+\phi)S_c^I = -(1+\phi)h_s S_s/h_c$ where ϕ is some number satisfying, say, $-0.1 \le \phi \le 0.1$. Then we have

$$\mathcal{N} = -\phi h_s S_s \neq 0. \tag{A.27}$$

Under these circumstances it may be feasible to apply a pressure difference to null out the displacement, in which case the condition (A.22) for no deformation becomes

$$\mathcal{N} - 2aF^{\#}p = 0, \tag{A.28}$$

neglecting gravity. According to (A.27) and (A.28), applying a pressure difference of

$$p = \frac{\mathcal{N}}{2aF^{\#}} = -\phi\frac{h_s S_s}{2aF^{\#}} \tag{A.29}$$

will thus maintain the initial parabolic shape. For a one-meter diameter $f/2$ mirror with membrane CTE mismatch stress 21.754 MPa, as in (A.18), and undercompensated by a coating stress that is 90% of the prescribed value ($\phi = -0.1$), the required pressure difference would be very small: $p = 10.9$ Pa, or about 0.0016 psi.

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